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Exploring Anisotropic Expansion in Bianchi Type III Cosmological Models in Closed Spatial Geometries: Insights from Christoffel Symbols and Dingle Methods

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ABSTRACT

In this study, we explore the Bianchi type-III cosmological model using Christoffel's and Dingle method. The essential framework of the model is that to investigate the dynamics of the closed universe $k = 1$ and to examine curvature and geodesic motion properties specific to the Bianchi Type III configuration. Using, Christoffel symbols and Dingle's techniques, we investigate the equations that control the development of these models, emphasizing how the Christoffel symbols help us comprehend the gravitational interactions between matter and energy distributions. This research provides insights into the implications of anisotropic models for the closed universe's structure formation, revealing how the interplay between geometric and physical properties influences cosmological evolution.

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Introduction

In recent decades, cosmological studies have thoroughly investigated the Bianchi Type III model under anisotropic conditions, providing important understanding of the universe's large-scale behavior [1]. In contrast to isotropic models that presume uniformity across all spatial dimensions, Bianchi Type

III geometry incorporates directional variations, enabling a more thorough examination of cosmic development, particularly in periods when perfect symmetry is not applicable, like the early universe [2]. Many researchers have explored this model to examine its behavior under different anisotropic influences [3]. These encompass scalar fields, commonly utilized to depict the inflation field that fuels inflation, alongside anisotropic fluid distributions and magnetic fields [4]. These factors create complexities in the Einstein field equations, resulting in distinct solutions that vary from those seen in isotropic cosmologies [5]. These solutions illuminate the effects of anisotropic pressures and stresses on the rate of expansion and curvature changes of the universe [6]. Specifically, research has explored the significance of open spatial curvature typical of Bianchi Type III models and its interaction with anisotropic factors [7, 8]. This affects our comprehension of how departures from isotropy could impact the development of large-scale structures and the anisotropies seen in the cosmic microwave background (CMB) [9]. Moreover, scientists have investigated the effects of anisotropic dark energy and various unconventional elements within this framework, broadening its significance to contemporary cosmology [10, 11]. Utilizing both analytical and numerical approaches, which involve instruments like Christoffel symbols for metric tensor calculations and sophisticated methods for resolving differential equations, these studies have deepened our comprehension of the universe's dynamics [12]. The findings from these studies enhance theoretical cosmology and provide possible observational signatures that can be validated against present and upcoming astrophysical data, positioning the Bianchi Type III model as a crucial focus in cosmological research [7, 8].

Through these investigations, the Bianchi Type III model has emerged as a vital framework for understanding the complexities of the universe beyond the simplifying assumptions of isotropy. Its capacity to incorporate directional variations and account for anisotropic influences offers crucial insights into early cosmic evolution, large-scale structure formation, and the origins of observed anisotropies in the cosmic microwave background. As future observational data continues to refine our understanding of the cosmos, the theoretical advancements facilitated by the Bianchi Type III model are poised to remain integral to the broader field of cosmology.

Mathematical Framework of the Bianchi Type III Model

The study of the Bianchi Type III model under anisotropic conditions requires solving the Einstein field equations, which connect space-time geometry to energy and matter distributions through the Ricci tensor $R_{\mu\nu}$, and the energy-momentum tensor $T_{\mu\nu}$. In this analysis, we employ Dingle's method to systematically compute the Christoffel symbols and other geometric components required to analyze the model. Dingle's method provides an efficient approach to derive the non-zero Christoffel symbols from the metric tensor, simplifying the otherwise complex calculations. This framework enables the determination of the Ricci tensor, Einstein tensor, and their evolution in an anisotropic universe with matter and energy contributions. The following analysis applies Dingle's method to explore the dynamics of anisotropic expansion in the Bianchi Type III model, highlighting the directional dependence of scale factors and their implications for cosmological evolution. The Bianchi type-III cosmological model has long served as a fundamental framework for understanding the universe's large-scale structure and dynamic evolution, so that better understanding the general or metric form (ds^2) of the universe according to the Bianchi type-III model as-

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)e^{2x}dy^2 + c^2(t)dz^2 \quad (1),$$

t = The cosmic time,

$a(t), b(t), c(t)$ = The scale factors along the direction of x, y, z resceptively,

e^{2x} = The anisotropic and hyperbolic nature of the geometry and only dependence on x-axis. According to Minkowski, the metric term have three conditions as, when $ds^2 = 0$, then it is indicating light like interval, when $ds^2 > 0$, indicating space like interval and $ds^2 < 0$, it is indicating time like interval. The investigation of Bianchi type III cosmological model under anisotropic conditions requires solving the Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (2),$$

In the equation (2), left side part tells that, matter-energy how to curve space-time, and right side part tells that matter-energy how to move through the curved space-time.

Here, $R_{\mu\nu}$ is called as Ricci curvature tensor,

$g_{\mu\nu}$ is metric tensor of the form,

R is the scaler curvature,

G is the Universal gravitational constant,

c is the speed of light and

$T_{\mu\nu}$ is known as Stress-Energy tensor. The metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & b^2(t)e^{2x} & 0 \\ 0 & 0 & 0 & c^2(t) \end{pmatrix} \quad (3a),$$

$$\text{Or } g_{\mu\nu} = diag\left(-1, a^2(t), b^2(t)e^{2x}, c^2(t)\right) \quad (3a`).$$

The inverse energy-momentum tensor can be written as-

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{b^2(t)e^{2x}} & 0 \\ 0 & 0 & 0 & \frac{1}{c^2(t)} \end{pmatrix} \quad (3b),$$

$$\text{Or } g^{\mu\nu} = diag\left(-1, \frac{1}{a^2(t)}, \frac{1}{b^2(t)e^{2x}}, \frac{1}{c^2(t)}\right) \quad (3b`).$$

And the Stress-Energy tensor as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P + \pi_{\mu\nu} \quad (4),$$

Here, ρ, P are the energy density and isotropic pressure in the universe, respectively,

μ_μ are the four-velocity of the fluid element,

$\pi_{\mu\nu}$ is the Anisotropic stress tensor.

The $T_{\mu\nu}$ tensor can be written as

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{bmatrix} \quad (5),$$

where, P_x, P_y, P_z are the directional pressures, which are linked up with the scale factors of $a(t), b(t), c(t)$ and the anisotropic stress-energy tensor $\left(\pi_{\mu\nu}\right)$ can be written as

$$\pi_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \pi_x & 0 & 0 \\ 0 & 0 & \pi_y & 0 \\ 0 & 0 & 0 & \pi_z \end{bmatrix} \quad (6),$$

and also written as $\pi_{\mu\nu} = diag\left(0, \pi_x, \pi_y, \pi_z\right)$.

So the relationship between the $\pi_{\mu\nu} - P$ as

$$\pi_x = P_x - P, \quad (6a),$$

$$\pi_y = P_y - P, \quad (6b),$$

$$\pi_z = P_z - P \quad (6c).$$

So the average isotropic pressure will be $P = \frac{1}{3}\left(P_x + P_y + P_z\right)$. If the pressures along the directions differ significantly, $\pi_{\mu\nu}$ will have nonzero components. Now from the above equations (6a-6c), we got,

$$\pi_x = P_x - P = P_x - \frac{1}{3}\left(P_x + P_y + P_z\right) \quad (6a`),$$

$$\pi_y = P_y - P = P_y - \frac{1}{3}\left(P_x + P_y + P_z\right) \quad (6b`),$$

$$\pi_z = P_z - P = P_z - \frac{1}{3}\left(P_x + P_y + P_z\right) \quad (6c`).$$

If the directional pressure will be $P_x = P_y = P_z$ then the universe will be isotropic, $\pi_{\mu\nu} = 0$. Now calculating the resultant separated values of the Energy-Momentum tensor as

$$T_{00} = \rho + \pi_{00} \quad (6d),$$

$$T_{11} = a^2(t)P + \pi_{11} \quad (6e),$$

$$T_{22} = b^2(t)e^{2x}P + \pi_{22} \quad (6f),$$

$$T_{33} = c^2(t)P + \pi_{33} \quad (6g),$$

Now, applying the Dingle method to compute the Christoffel symbols of non-zero values and other more components required to analyze the model. The Christoffel symbols written as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left[\partial_\nu g_{\alpha\mu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right] \quad (7),$$

Here, in the equation (5), $\Gamma^{\lambda}_{\mu\nu}$ is the second kind Christoffel symbols and $g^{\lambda\alpha}$ is the inverse metric tensor $\left(g^{\lambda\alpha}g_{\lambda\nu} = \delta^{\lambda}_{\nu}\right)$, $\partial_{\nu,\mu,\alpha}$ is the partial derivate $\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\alpha}\right)$, respectively.

Components of Christoffel symbol:

From the equation (5), got some result for the non-zero Christoffel symbol for time (t) and spatial coordinates (x, y, z) dependencies,

$$\Gamma^{0}_{11} = \frac{1}{2}(-1)\partial_0(a^2) = -a\dot{a} \quad (8a),$$

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{\dot{a}}{a} \quad (8b),$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{b}}{b} \quad (8c),$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1 \quad (8d),$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{c}}{c} \quad (8e),$$

$$\Gamma^0_{22} = b\dot{b}e^{2x} \quad (8f),$$

$$\Gamma^0_{33} = c\dot{c} \quad (8g),$$

$$\Gamma^1_{22} = -e^{2x}\frac{b^2}{a} \quad (8h),$$

This terms are involving the time derivatives of scale factors, describing expansion or contraction in every directions of x, y, z and the exponential term, the dot and double dot over the scale factors are indicating the differentiation and double differentiation with respect to time.

Components of Ricci Tensor and Scaler Tensor:

From the equation (2), the value of $R_{\mu\nu}$ calculated as given below,

$$R_{00} = -\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) \quad (7a),$$

$$R_{11} = \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2} \quad (7b),$$

$$R_{22} = \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2} \quad (7c),$$

$$R_{33} = \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} \quad (7d).$$

Here, k is the spatial curvature parameter. It is determine the shape of the universe as

If $k = 0$, then the universe will be flat or no spatial curvature (Euclidian),

If $k > 0$, then the universe will be closed or positively curved (Spherical),

If $k < 0$, then the universe will be Open or negatively curved (Hyperbolic). (Here we focus only in the open or hyperbolic structural model.) Here, $\ddot{a} = \frac{\partial^2 a}{\partial t^2} = \frac{\partial \dot{a}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\right)$, and similarly as for the $\ddot{b}$ and $\ddot{c}$.

So the relation between the Ricci tensor and Scaler tensor can be written as

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \quad (8),$$

From the equation (8), the Scaler tensor R can be calculated as

$$R = \left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) + \frac{1}{a^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2}\right) + \frac{1}{b^2}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right) \quad (8a).$$

From the equations 7(a)-7(d), (8) and (6d-6g), we got

$$R_{00} - \frac{1}{2}g_{00}R = \frac{8\pi G}{c^4}T_{00}$$

$$\frac{1}{2}\left[-\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) + \frac{1}{a^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2}\right) + \frac{1}{b^2}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right)\right] = \rho + \pi_{00} \quad (9a),$$

$$R_{11} - \frac{1}{2}g_{11}R = \frac{8\pi G}{c^4}T_{11}$$

$$\frac{1}{2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2}\right) - \frac{a^2}{2}\left[\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) + \frac{1}{b^2}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right)\right] = a^2P + \pi_{11} \quad (9b),$$

$$R_{22} - \frac{1}{2}g_{22}R = \frac{8\pi G}{c^4}T_{22}$$

$$\frac{1}{2}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2}\right) - \frac{b^2}{2}\left[\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) + \frac{1}{a^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right)\right] = b^2e^{2x}P + \pi_{22} \quad (9c),$$

$$R_{33} - \frac{1}{2}g_{33}R = \frac{8\pi G}{c^4}T_{33}$$

$$\frac{1}{2}\left(\frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right) - \frac{c^2}{2}\left[\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) + \frac{1}{a^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2}\right) + \frac{1}{b^2}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2}\right)\right] = c^2P + \pi_{33} \quad (9d).$$

From the derived results, we have successfully obtained the directional Einstein field equations for an anisotropic and homogeneous cosmological model. In the context of cosmology and dynamical systems, the use of power-law expressions to describe acceleration or deceleration parameters provides an effective framework for understanding how these quantities evolve over time, distance, or other variables.

The Power Law

To explore the acceleration or deceleration of the universe, a methodical approach rooted in cosmology and dynamical systems is essential. This involves analyzing the behavior of scale factors, energy conditions, and effective equations of state, offering insights into the dynamics of the universe and the factors driving its expansion or contraction.

With power-law, we can express the relationship as

Here, we assume that the scale factor $a(t) = b(t)$ are evolve same and $c(t)$ evolve differently, so the power law can be written as

$$a(t) = b(t) = t^m \quad \text{(10a)},$$

$$c(t) = t^n \quad \text{(10b)},$$

In the equation (10a) and (10b), the m, n are the power law exponents, which determines the rate of universe expansion or contraction. The m is governs the time dependence of scale factors $a(t), b(t)$ and n describing the time evolution of scale factor $c(t)$, which describe the contraction or expansion along the direction of $x - y$ and z axis.

The power law exponents having few conditions that are

- If the expansion rate is anisotropic then it follows must be $m \neq n$, and
- If the expansion rate of the universe is isotropic the it follows the $m = n$.

After some calculations, the results are

$$\dot{a} = \dot{b} = mt^{m-1} \text{ and } \ddot{a} = \ddot{b} = m(m-1)t^{m-2} \quad \text{(11a)},$$

$$\dot{c} = nt^{n-1} \text{ and } \ddot{c} = n(n-1)t^{n-2} \quad \text{(11b)},$$

Now the equations 11(a,b) are putting gin the equations 7(a-d), the resultant values of Ricci tensor are given below

$$R_{00} = -\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) = -\frac{1}{t^2}[2m(m-1) + n(n-1)] \quad \text{(12a)},$$

$$R_{11} = \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{c}\dot{a}}{ca} - \frac{k}{a^2} = \frac{1}{t^2}\left[\frac{2m^2 + m(n-1) - kt^2}{t^{m+1}}\right] \quad \text{(12b)},$$

$$R_{22} = \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} - \frac{ke^{2x}}{a^2} = \frac{1}{t^2}\left[\frac{2m^2 + m(n-1) - kt^2e^{2x}}{t^{m+1}}\right] \quad \text{(12c)},$$

$$R_{33} = \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} = \frac{n(n-1)+2mn}{t^2} \qquad (12d),$$

Now taking the equation (12a) and (12d), since the cosmic time $t > 0$, then we neglecting $\frac{1}{t^2}$, and after few algebraic calculation, we got the values for the m, n as $1, 0$, respectively. From the values of m, n, here we apply to calculate the value of curvature parameter using equation (12b), after some algebraic calculation we got $k = 1$. The explanation of the value of k as

- For the value of $k = 1$, the universe shape become the closed or positively curvature (spherical), which we have been chosen in our study.

Now putting the value of m, n in the equations (12a-12d),

$$R_{00} = 0,$$

$$R_{11} = \frac{1}{t^4}(1 - t^2),$$

$$R_{22} = \frac{1}{t^4}(1 - t^2 e^x),$$

$$R_{33} = 0 \qquad (13)$$

From the above results of equations (13a-13d), can be explain as

- R_{00} is the related to time component and linked with the acceleration or deceleration of the universe evolution
- R_{11} is the curvature dynamically changes over time. For small ttt, the curvature is positive, but as time progresses, it transitions, which could indicate an evolution in the expansion behavior.
- R_{22} is the curvature in this direction evolves differently compared to R_{11} , reinforcing the anisotropic nature of the expansion.
- R_{33} is the related to the z- directional curvature, from the results there no expansion along the z-direction.

From the above calculation using power law, we got quite satisfying results for m, n values. Now putting the values of m, n in the equation of power law (10a, 10b), then we got the result as

(10a), $a(t) = b(t) = t^m = t^1 = t$.

(10b), $c(t) = t^n = t^0 = 1$.

Here we distribute the different values of m, n in different cases,

Special case:

$\boldsymbol{m = 1}$ & $\boldsymbol{n = 0}$

- The universe expands linearly in two directions (x and y).
- The third direction (z) remains static ($c = 1$), meaning no expansion or contraction along this axis.
- This represents anisotropic expansion, where space evolves only in two dimensions while remaining unchanged in the third.
- Physically, this could describe a universe constrained in one dimension, possibly influenced by external conditions like an anisotropic energy distribution.

Graphical Representation

The graph illustrates the anisotropic expansion of the universe with scale factors $a(t) = b(t) = t$ and $c(t) = 1$. The blue and red lines show that expansion occurs linearly in two directions, while the green line indicates that the third direction remains static. This suggests a pancake-like expansion, where the universe evolves in two dimensions while remaining unchanged in the third. Such behavior is relevant to Bianchi Type III models, highlighting directional dependence in cosmological evolution. The results demonstrate that expansion is not uniform, supporting the presence of anisotropic influences in the universe's dynamics.

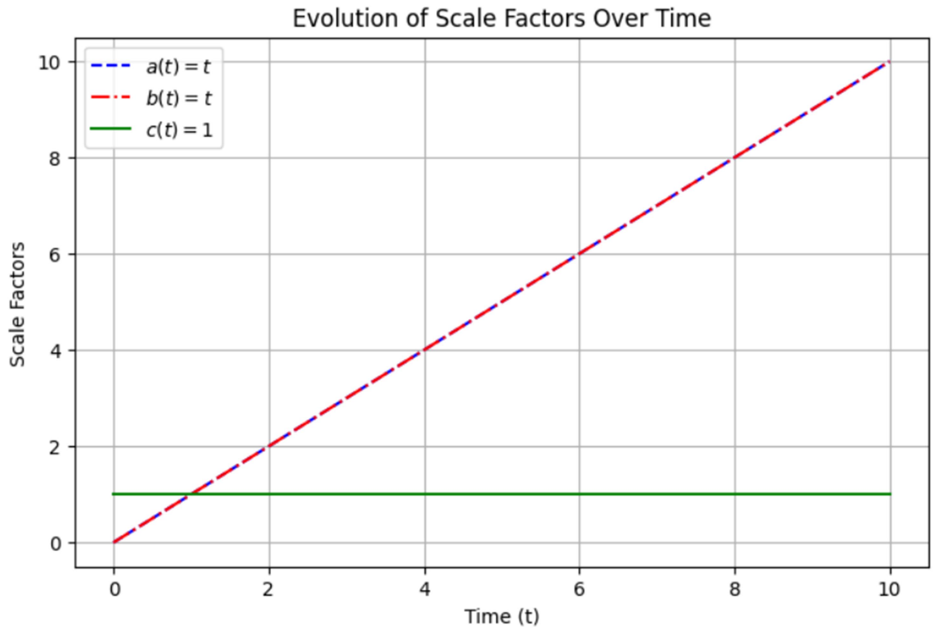

Fig: Anisotropic expansion of closed universe, scale factor vs time

Now we are ready to conclude the study which is given below.

Conclusion of the study

In this study, we investigated the dynamics of Bianchi Type III cosmological models under anisotropic conditions in a closed, spherical universe. Employing Dingle's method, we derived key geometric components and solved Einstein's field equations to analyze how anisotropy influences cosmic evolution. Our findings reveal that directional variations in pressure lead to expansion behaviors distinct from isotropic models. Through a power-law approach to scale factors, we demonstrated how expansion rates differ across spatial directions, with the curvature parameter $k = 1$ confirming the universe's closed, positively curved nature. This study enhances our understanding of anisotropic expansion and its role in large-scale structure formation, offering valuable insights for refining cosmological models as observational data advances.

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