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WARPED PRODUCT CR AND SEMI SLANT SUBMANIFOLDS WITHIN THE FRAMEWORK OF QUASI PARA SASAKIAN MANIFOLDS

Shamsur Rahman [1], Sabi Ahmad *[2], and Niranjan Kumar Mishra [3]
[1] Maulana Azad National Urdu University, Polytechnic Satellite Campus Darbhanga Bihar- India.
[2] Rkdf , University Ranchi, Jharkhand 834004, India, ahmadsabi361@gmai1.com,
[3] Rkdf, University Ranchi, Jharkhand, India

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ABSTRACT

The purpose of the present paper is to study the notion of warped product CR and semi slant submanifolds with a quasi paraSasakian. The existence of such warped product of the types M $=N_T \times_\delta N_\theta$ and = NL in quasi para Sasakian manifold is shown some interesting results. We establish fundamental properties, derive characterization theorems, and explore conditions for the existence of such submanifolds. Relationships between the geometry of the submanifold and the ambient quasi-para-Sasakian structure are analyzed. Several examples are provided to illustrate the theoretical results. Our findings extend and generalize known results in the study of warped product submanifolds in almost contact geometry.

1 INTRODUCTION

The notion of semi-slant submanifolds of almost Hermitian manifolds has been introduced by N. Papaghuic [9]. After that the concept has been considered by by J. L. Cabrerizo et al. [5] for almost contact metric manifolds. The notion of CR submanifolds in a Kahler manifold was defined and studied by A. Bejancu [1]. Moreover Blair [4] and Kanemaki [Il] have defined and studied quasi Sasakian manifold and proved some properties of contact CR —submanifolds. Also Bishop and O'Neill [101 introduced the notion of warped product manifolds. These manifolds appear in differential geometric studies in a natural way. XVarped product submanifolds of Kaelller manifolds was introduced by B. Y. Chen [3]. Later on, B. Sallin [2] extended the results of Chen's for warped product semi-slant submanifolds of Kaelller manifolds. K. A. Khan et al. studied warped product semi-slant submanifolds in cosymplcctic manifolds and proved that there exist no proper warped product semi slant submanifolds in the form and reversing the two factors in cosymplectic manifolds [6]. Siraj Uddin et al. studied warped product semi-slant submanifolds of a Kenmotsu manifold [14]. In [12, 13] Authors studied NVarped Product slant immersions in quasi Sasakian manifolds. In M. Atcseken proved that the warped product submanifolds of the types M = and M = No of a Kenmotsu manifold do not exist where the manifolds NO and NT (resp., N^L are proper slant and -invariant (resp. anti-invariant) submanifolds of a Kenmotsu manifold M, respectively [8].

In present paper, we study the warped product of the types M = NT and M = NL and obtain characterization for warped product submanifolds in terms of warping function and shape operator.

2 PRELIMINARIES

Let M be a (2n + l)-dimensional almost paracontact manifold with structure tensor v, < , >), where f, and v be a tensor field of type (1, 1), a vector field, and a 1-form respectively on satisfying

$$f\xi = 0, \quad f^2 = I - v \otimes \xi, \quad v \circ f = O \quad (1)$$

$$v(\xi) = 1, \quad v(\mathcal{X}) = <\mathcal{X}, \xi>$$
$$(2) \quad <f\cdot, f\cdot> = -<,> + v \otimes v$$

where I is the identity on the tangent bunble TM on M. We say that $\bar{M}$ is a paracontact metric manifold if there exists a one-form v such that [15]

$$< X, f)' >= dv(= \tfrac{1}{2}(\mathcal{X}v(\mathcal{Y}) - Yv(X) - v([\mathcal{X}, \mathcal{Y}])\ \forall x,y \in XM$$

and

$$<f\mathcal{X}, \mathcal{Y}> + <x, fy> = 0 \quad (3)$$

for all vector fields X and y on M. Further, an almost paracontact metric manifold is called a quasi para Sasakian manifold if

$$(\nabla_X f)Y = v(\mathcal{Y})\mathcal{F}\mathcal{X} - <FX,Y> \xi \quad (4)$$

and

$$\nabla_{\mathcal{X}} \xi = -fFX, \qquad = FfX \quad (5)$$

where V denotes the Levi-Civita connection with respect to the metric tensor

Apply f to (5) and using (l) obtain

$$\mathcal{F}\mathcal{X} = v(FX) - f(\nabla_X \xi) \quad (6)$$

Also, replace X by in (5) and by using second relation, wc get

(7)

Using (4), (6) and (7) infer

$$\mathcal{F}\xi = v(\mathcal{F}\xi)\xi \quad (8)$$

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and

$(\bar{\nabla}_\xi f)\mathcal{X} = 0$(9) for any X G r (TM).

For a submanifold M of a Riemannian manifold M, the formulas of Gauss and Weingarten are given respectively by

(10) $$\bar{\nabla}_\mathcal{X}\mathcal{Y} = \sigma(\mathcal{X},\mathcal{Y}) + \nabla_\mathcal{X}\mathcal{Y}$$

(11) $$\bar{\nabla}_\mathcal{X}\lambda = -\Lambda_\lambda\mathcal{X} + \nabla^\perp_\mathcal{X}\lambda$$

for X, y G TM and for normal vector field N of M, where V is the induced Levi-Civita connection on M, the second fundamental form, V the normal connection, and A the shape operator. The shape operator and the second fun(lamental form of M are related by

$$< \sigma(\mathcal{X},\mathcal{Y}), \lambda >=< \Lambda_\lambda\mathcal{X}, \mathcal{Y} >$$ 421b

where < , > denotes the induced metric on M as well as the metric on f. If M is a contact CR-submanifold of M and the projections on D and D ^{L} by P and Q respectively, then for all vector field X tangent to M, we infer

X = PX + QA' + (13)

Now put

$BN + CN = fN$

where BN and CN are tangential and normal part of fN on M.
Next we define the tensor field of type (1, 1) on M by

f,l' = fP,1' (15)

and the r (TJVt)- valued -form w by

wX = fQ,1' (16)

Since D is invariant by f, it follows from (14) and (15) that B is r (D ^{L})-valued and t is T(D)-valued respectively.
By using (1), (13), (15) and (16), obtain

$$\omega\mathcal{X} + t\mathcal{X} = f\mathcal{X} \quad (17)$$

and

$$t^3 + t = 0; C^3 + C = 0 \quad (18)$$

Then by (18) we conclude that t and C arc f-structurc in scnsc K. Yano [7] on TM and "I'.M ^{L} respectively.
Further on, for any Z e T (TM), wc put

$$\mathcal{F}\mathcal{Z} = \alpha\mathcal{Z} + \beta\mathcal{Z} \quad (19)$$

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where and ßZ are the tangent part and the normal part of FZ respectively. From (3) and (17) we have

< + < x,ty > o (20)

Taking account of (6), (10), (14) and (17) obtain

(21)

ax = $\nu(\mathcal{X})v(\mathcal{F}\mathcal{X})\xi$— — BIE(X, E)

and

$$\beta\mathcal{X} = -\omega(\nabla_{\mathcal{X}}\xi) - Ch(\mathcal{X},\xi) \quad (22)$$

Proposition I. If M is a contact CR-submanifold of a quasi para-Sasakian manifolds M, then r (T'M) is invariant with respect to the action of f if and only if we have w(Vxs^{C}) = o(23)

and

$$Ch(\mathcal{X},\xi) = 0 \quad (24)$$

PROOF. From (22) it follows that F is a tensor field of type (I, I) on M if and only if 4/15

(25)

Then (23) and (24) follows from (25) since < WY, O for any y r(T-'VI)

Corollary I. is a contact CR-submanifold of a quasi pam-Sasakian manifolds such that r (TM) is invariant with respect to the action of F, then both distribution D and D ^{L} are invariant with respect to the action of F.

PROOF. Consider X T (D) and by using the third relation in (4) and (8) obtain

$<\mathcal{F}\mathcal{X},\xi>=$ - < x, v(FÇ) $<\mathcal{X},\xi>$

On the other hand, by using (2), the second relation in (4) and the invariace of D with respect to the action of f we infer

< EX, z FfX',Z — < EX',fZ O

where X' e T (D) and Z r (D^{L}). Hence D is invariant by F. In a similar way it follows that D ^{L} is invariant by the action of F.

The Riemannian connections V and V ^{L} allow us to define as usually the covariant derivatives

(Vxt)Y = VxtY - tVxY (26)

and

(Vxw)Y = vxwy - wVxY (27)

Now, the canonical structures t and w on a submanifold are said to be parallel if Vt = 0 and Va.' = 0, respectively. On a submanifold of a quasi paraSasakian manifold by equations (5) and (10), wc get

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$$\nabla_{\mathcal{X}}\xi = -fFX \quad (28)$$

and

$$\sigma(\mathcal{X},\xi) = 0 \quad (29)$$

for each X e "I'JVf . Furthermore, from equation (29)

$$v(\Lambda_\omega)\mathcal{X} = 0 \quad (30)$$

Lemma 1. If M is a contact CR-submanifoId of a quasi para-Sasakian manifolds M, then we have

(Vxt)Y = Awyx + 3 $h(\mathcal{X},\mathcal{Y}) + v(\mathcal{Y})\alpha\mathcal{X} -$ < EX, Y >ξ (31)

(Vxw)Y = $\sim h(\mathcal{X},\mathcal{Y})$ - h(X, tY) +$v(\mathcal{Y})\alpha\mathcal{X}$ (32)

PROOF. By using (4), (26) and (27) obtain (ax + < EX, Y > = - AwyX- tY) for any X, y r (TJVI). Then (31) and (32) follows the above identity by identifying the tangent parts and the normal parts respectively. The covariant derivatives Of B and C arc given by

(vxB)N = VxBN -13(vkN) (33)

and

(vk.c)N = vkCN - c(vkN) 5/15

respectively, for any X G T (TJVt) and N r (TM I).

Lemma 2. If M is a contact CR-submanifoId of a quasi para-Sasakian manifolds M, then we have

(VxB)N = AcxX - t(AMX)- < rx,N >ξ (35)

and

(vkc)N = $-h(\mathcal{X}, B\mathcal{N})$ - w(ANX) (36)

for any X e T(TM) and N T(TM L).

Lemma 3. If M is a contact CR-submanifold of a quasi para-Sasakian manifolds M. Then wc have

11fX y =$\Lambda_{f\mathcal{Y}}\mathcal{X}$ (37)

and

< h(u, $\mathcal{V}), f\mathcal{Z}$ >=< VuZ,fV > (38)

for any u e $\Gamma(T\mathcal{M}), \mathcal{V} \in$ I'(D) and X, Y, Z e T(D L).

G

PROOF. By using (2), (4) and (10) -(12), we infer

< AjxY,U $\mathcal{Y},\mathcal{U}$ >=< $h(\mathcal{Y},\mathcal{U}), f\mathcal{X}$ >=< VuY,fX > − < $\nabla_{\mathcal{U}}\mathcal{Y}, f\mathcal{X}$ >

$\mathcal{V}, f\mathcal{X}$ >= − < $f(\nabla_{\mathcal{U}}\mathcal{Y}), \mathcal{X}$ >= − < $-(\bar{\nabla}_{\mathcal{U}} f)\mathcal{Y} + \bar{\nabla}_{\mathcal{U}} f\mathcal{Y}, \mathcal{X}$ >

=< VuY,fX + < $v(\mathcal{Y})\mathcal{F}\mathcal{U}-$ < $\mathcal{F}\mathcal{U},\mathcal{Y}$ > $\xi, \mathcal{X}$ > − < $\bar{\nabla}_{\mathcal{U}} f\mathcal{Y}, \mathcal{X}$ >

$-\Lambda_{f\mathcal{Y}}\mathcal{U} + \nabla^{\perp}_{\mathcal{U}} f\mathcal{Y}, \mathcal{X}$ >=< AfYU,X >=< AfYX,U >.

- < -AfYU + våfY,x

since v(Y) = v(X) = O. Thus we have (2.37) Next, by using (2), (4) and (10) obtain

< IE(u, $\mathcal{V}), f\mathcal{Z}$ >=<VuV,fZ > - < V, VufZ >

< V, (Vuf)Z + f(VuZ)) > — < < FIR, z >

< V, f(VuZ) p, Vuz fU,VuZ >

A submanifold M of an almost para contact metric manifold M is said to be invariant if F is identically zero, that is, f X T"VI and anti-invariant if t is identically zero, that is, fX G T I JVf, for any X G TM.

For each non zero vector X tangent to M at a: such that X is not proportional to (, wc denotes by O(X), the angle between fX and TrJVt for all a: M.

Definition 1. M is said to be slant if the angle ()(X) is constant for all X ∈
TxJVf — and a: M. The angle 0 is called a slant angle or Wirtinger angle. Obviously, if 0 = O, is invariant and if 0 = T/ 2, M is an anti-invariant submanifold. If thc slant angle of M is different from O and T1/2 then it is called proper slant.

A characterization of slant submanifolds is given by the following theorem.

Theorem I. Let N bc submanifold of a quasi para Sasakian manifold M such that is tangent to N. Then N is slant submanifold if and only if there exists a constant [O, I] such that

$$t^2 x = \quad — \qquad (39)$$

Furthermore, if O is the slant angle of N , then = $\cos^2 0$

From (39), we get

< tZ,tW(40) < $>= \cos^2\theta\{-<\mathcal{Z},\mathcal{W}>+v(\mathcal{Z})v(\mathcal{W})\}$ nZ,nW(41) for any Z, W tangent to N. $>= \sin^2\theta\{-<\mathcal{Z},\mathcal{W}>+v(\mathcal{Z})v(\mathcal{W})\}$

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