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Maximum Degree Energy of Globe Graph, Bistar Graph and Some Graph Related to Bistar Graph

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ABSTRACT

The graphs considered in this article are undirected, finite and simple graphs. In this article we have proved that Maximum Degree energy of Bistar graph $B_{n,n}$ is $4(n+1)\sqrt{4n+1}$. Also we have investigated Maximum Degree energy of some graphs related to Bistar graph and Globe graph.

1. Introduction

The energy of a graph introduced by Gutman [6] is an important concept of spectral graph theory which links Organic Chemistry to linear algebra of Mathematics. Generally, graph's energy is summation of absolute values of eigenvalues of the adjacency matrix. Similar energies got from the eigenvalues of various graphs are considered in recent times. Several other authors also investigated energy of graph [2,7,8,9,10,11].

In a graph G, the distance between any two vertices u and v is denoted as $d(u,v)$ and it is defined as the length of the minimum path connecting them, if there is no path between u and v then $d(u,v)$ is defined as ∞. It is useful to note that in a connected graph distance between any two vertices is always finite provided graph is finite. For a vertex v of G, the Eccentricity of a vertex v is denoted as $e(v)$ and is defined as $e(v) = max\{d(u,v); u \in V(G)\}$.

In [1] C. Adiaga and M. Smitha introduced the concept of Maximum Degree matrix $M(G)$ of connected graph G and it is defined as,

$$d_{ij} = \begin{cases} max\{d(v_i), d(v_j)\}, & If\ v_iv_j \in E(G); \\ 0, & Otherwise \end{cases}$$

The Maximum Degree eigenvalues are the eigenvalues of matrix $M(G)$.

In this article we have considered only finite, simple undirected graphs. It is useful to recall some definitions from graph theory.

Definition 1.1 [1]: The *Maximum Degree energy* of graph G is denoted as $E_M(G)$ and it is defined by $E_M(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \lambda_3, \dots \dots \lambda_n$ are the eigenvalues of $M(G)$. Where $M(G)$ is the Maximum Degree matrix of graph G.

Definition 1.2 [3]: A graph G is said to be *bipartite* if the vertex set V of G can be partitioned into two disjoint subsets $V_1$and V_2 such that $V_1 \cap V_2 = \emptyset$ and for each edge has one end vertex is in $V_1$and other is in V_2.

Definition 1.3 [3]: A *complete bipartite* graph is a bipartite graph in which all the vertices in V_1 are adjacent with all the vertices of V_2. If $|V_1| = m$ and $|V_2| = n$ respectively then the corresponding complete bipartite graph is denoted as $K_{m,n}$.

Definition 1.4 [3]: A complete bipartite graph $K_{1,n}$ is known as star graph. Here the vertex of degree n is called the apex vertex.

Definition 1.6 [5]: *Bistar* $B_{n,n}$ is the graph obtained by joining the centre (apex) vertices of two copies of $K_{1,n}$ by an edge. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{v_1, v_2, \dots \dots, v_n, v, u, u_1, u_2, \dots \dots u_n\}$, where v, u are apex vertices and $v_1, v_2, \dots \dots, v_n, u_1, u_2, \dots \dots u_n$ are pendent vertices. The edge set of $B_{n,n}$ is $E(B_{n,n}) = \{vv_1, vv_2, \dots \dots, vv_n, vu, uu_1, uu_2, \dots \dots uu_n\}$.

Definition 1.7 [11]: For a simple connected graph G the *square of graph* G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 1.8 [11]: The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' ad G'' and join each vertex u' in to G' the neighbours of the corresponding vertex v' in G''.

Definition 1.9 [9]: A *globe graph Gl(n)* is a graph obtained from two isolated vertex are joined by n paths of length two.

Definition 1.10 [11]: For a graph G the *splitting graph* $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$

In theorem 2.1, we have shown that Maximum Degree energy of Bistar graph $B_{n,n}$ is $4(n+1)\sqrt{4n+1}$. for $n \in N, n \neq 1$. We also provided supportive example in Example 2.2 and in that example we have prove that $E_M(B_{5,5}) = 24\sqrt{21}$. We also investigate the Maximum Degree of square of bistar graph in Theorem 2.3 and we have shown that its Maximum Degree energy is $(2n+1)(1+\sqrt{64n+1})$. In Theorem 2.5 we proved that Maximum Degree energy of shadow graph of bistar graph $D_2(B_{n,n})$ is $8(n+1)\sqrt{4n+1}$ and in Theorem 2.7 we proved that the Maximum Degree energy of Globe graph is $4\sqrt{2n}$.

2. Main Result

Theorem 2.1: Let $n \in N, n \neq 1$. Then $E_M(B_{n,n}) = 4(n+1)\sqrt{4n+1}$, where $E_M(B_{n,n})$ is the

Maximum Degree energy of graph $B_{n,n}$.

Proof: Let $V(B_{n,n}) = \{v_1, v_2, \dots \dots, v_n, v, u, u_1, u_2, \dots \dots u_n\}$. Note that $B_{n,n}$ is graph with $2n+2$ vertices and $2n+1$ edges as shown in the folowing Figure 1.

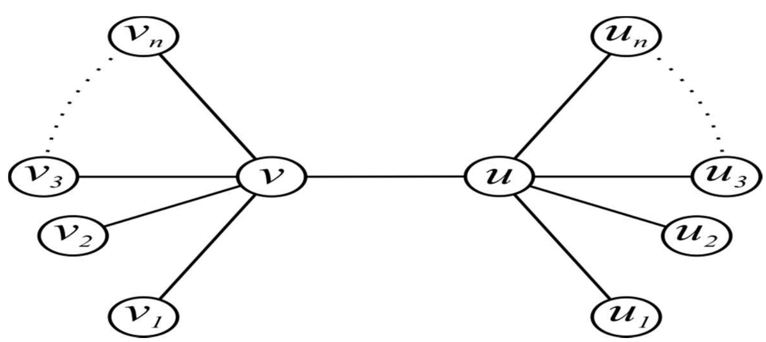

Figure 1 Bistar graph $B_{n,n}$

Observe that the Maximum Degree matrix $M(B_{n,n})$ of $B_{n,n}$ is given by

$$M(B_{n,n}) = \begin{array}{c} \\ v_1 \\ \vdots \\ v_n \\ v \\ u \\ u_1 \\ \vdots \\ u_n \end{array} \begin{bmatrix} v_1 & \cdots & v_n & v & u & u_1 & \cdots & u_n \\ 0 & \cdots & 0 & n+1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & n+1 & 0 & 0 & \cdots & 0 \\ n+1 & \cdots & n+1 & 0 & n+1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & n+1 & 0 & n+1 & \cdots & n+1 \\ 0 & \cdots & 0 & 0 & n+1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & n+1 & 0 & \cdots & 0 \end{bmatrix}$$

Note that the characteristic polynomial of matrix $M(B_{n,n})$ is $\lambda^{2(n+1)} - 4(2n^3 + 5n^2 + 4n + 1)\lambda^{2n} - 16n^2(n+1)^4\lambda^{2(n-1)}$

So, eigenvalues of $M(B_{n,n})$ are $0,0,\ldots\ldots,0(2(n-1)\ times), (n+1)(1+\sqrt{4n+1}), (n+1)(1-\sqrt{4n+1})$, $(n+1)(-1+\sqrt{4n+1})$ and $(n+1)(-1-\sqrt{4n+1})$.

Hence, Maximum Degree energy of $B_{n,n} = E_M(B_{n,n}) = 0 + |(n+1)(1+\sqrt{4n+1})| + |(n+1)(1-\sqrt{4n+1})| + |(n+1)(-1+\sqrt{4n+1})| + |(n+1)(-1-\sqrt{4n+1})|$

$= (n+1)(1+\sqrt{4n+1} - 1 + \sqrt{4n+1} - 1 + \sqrt{4n+1} + 1 + \sqrt{4n+1})$

$= 4(n+1)\sqrt{4n+1}$.

Example 2.2: Maximum Degree energy of Bistar graph $B_{5,5} = 24\sqrt{21}$

Proof:

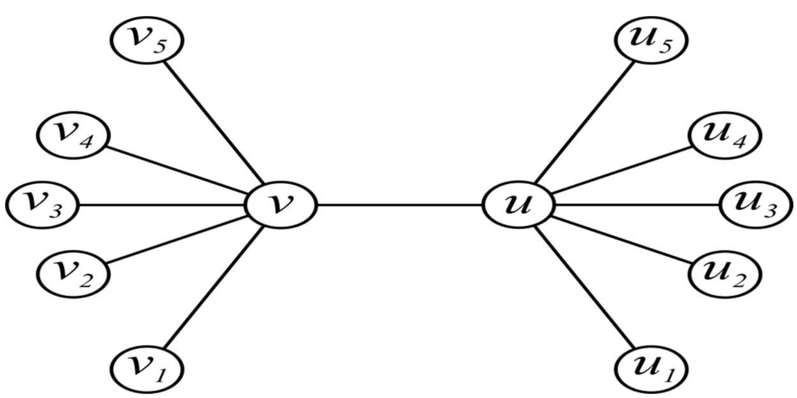

Figure 2 Bistar graph $B_{5,5}$

The Maximum Degree matrix of $B_{5,5}$ is given by

$$M\left(B_{5,5}\right) = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v & u & u_1 & u_2 & u_3 & u_4 & u_5 \\ v_1 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 6 & 6 & 6 & 6 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 6 & 6 & 6 & 6 \\ u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ u_5 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Note that the characteristic polynomial of matrix $M\left(B_{5,5}\right)$ is $\lambda^{12} - 1584\lambda^{10} - 518400\lambda^{8}$

So, Maximum Degree eigenvalues of $B_{5,5}$ are $0,0,\dots,0(8\ times), 6(\sqrt{21}+1), 6(\sqrt{21}-1), 6(-\sqrt{21}+1)$ and $6(-\sqrt{21}-1)$

Hence, Maximum Degree energy of $B_{5,5} = 0 + |\,6(\sqrt{21}+1)| + |\,6(\sqrt{21}-1)| + |6(-\sqrt{21}+1)| + |6(-\sqrt{21}-1)|$. Therefore, Maximum Degree energy of $B_{5,5} = E_M\left(B_{5,5}\right) = 24\sqrt{21}$

Theorem 2.3: Let $n \in N$, $n \neq 1$. Then $E_M\left(B_{n,n}^2\right) = (2n+1)\left(1+\sqrt{16n+1}\right)$, where $E_M\left(B_{n,n}^2\right)$ is the Maximum Degree energy of graph $B_{n,n}^2$.

Proof: Let $V\left(B_{n,n}^2\right) = \{v_1, v_2, \dots \dots, v_n, v, u, u_1, u_2, \dots \dots u_n\}$. Note that $B_{n,n}^2$is graph with $2n+2$ vertices and $4n+1$ edges as shown in the following Figure 3.

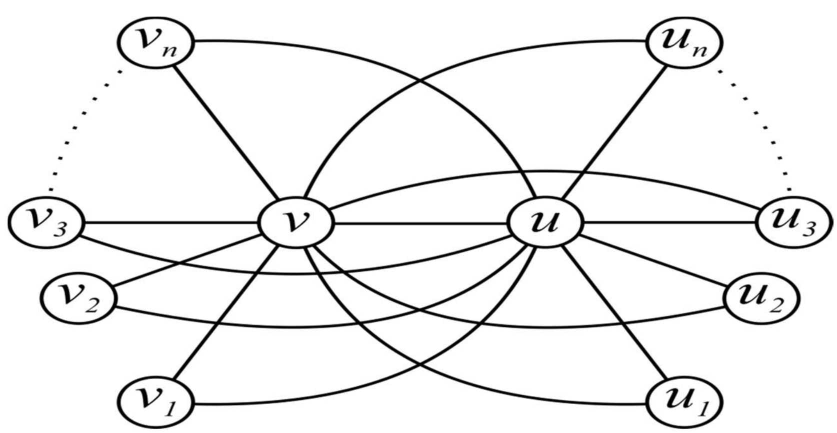

Figure 3 Square of Bistar graph $B_{n,n}^2$

Observe that the Maximum Eccentricity matrix $M\left(B_{n,n}^2\right)$ of $B_{n,n}^2$ is given by

$$M\left(B_{n,n}^2\right) = \begin{array}{c} \\ v_1 \\ \vdots \\ v_n \\ v \\ u \\ u_1 \\ \vdots \\ u_n \end{array}\begin{bmatrix} v_1 & \cdots & v_n & v & u & u_1 & \cdots & u_n \\ 0 & \cdots & 0 & 2n+1 & 2n+1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n+1 & 2n+1 & 0 & \cdots & 0 \\ 2n+1 & \cdots & 2n+1 & 0 & 2n+1 & 2n+1 & \cdots & 2n+1 \\ 2n+1 & \cdots & 2n+1 & 2n+1 & 0 & 2n+1 & \cdots & 2n+1 \\ 0 & \cdots & 0 & 2n+1 & 2n+1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n+1 & 2n+1 & 0 & \cdots & 0 \end{bmatrix}$$

Note that the characteristic polynomial of $M\left(B_{n,n}^2\right)$ is $\lambda^{2(n+1)} - (4n+1)(2n+1)^2\lambda^{2n} - 4n(2n+1)^3\lambda^{(2n-1)}$

So, Maximum Degree eigenvalues are $0,0,\ldots\ldots,0$ $((2n-1)\ \ times), -(2n+1)$,

$\frac{(2n+1)}{2}\left(1+\sqrt{16n+1}\right)$ and $\frac{(2n+1)}{2}\left(1-\sqrt{16n+1}\right)$.

Therefore, Maximum Degree energy of $B_{n,n}^2 = E_M\left(B_{n,n}^2\right) = 0 + |-(2n+1)| + \frac{(2n+1)}{2}\left|1+\sqrt{16n+1}\right| + \frac{(2n+1)}{2}\left|1-\sqrt{16n+1}\right|$

$$= 2n+1+\frac{(2n+1)}{2}\left(1+\sqrt{64n+1}-1+\sqrt{64n+1}\right)$$

$$= (2n+1)\left(1+\sqrt{16n+1}\right).$$

Example 2.4: Maximum Degree energy of square of Bistar graph $B_{5,5}^2 = 110$

Proof:

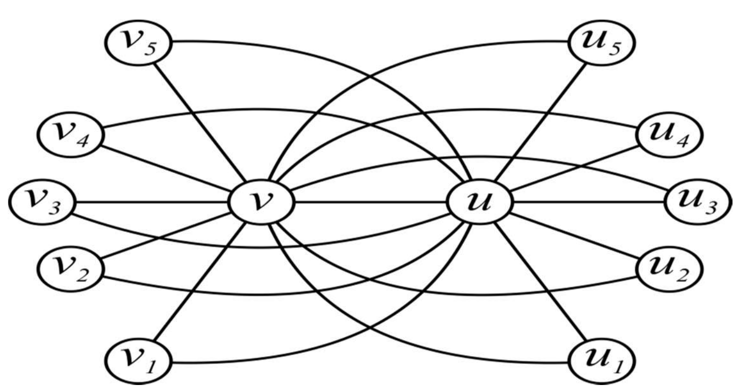

Figure 4 Square of Bistar graph $B_{5,5}^2$

The Maximum Degree matrix of $B_{5,5}^2$ is given by

$$M\left(B_{5,5}^{2}\right)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v\\ u\\ u_1\\ u_2\\ u_3\\ u_4\\ u_5\end{array}\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v & u & u_1 & u_2 & u_3 & u_4 & u_5\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 11&11&11&11&11&0&11&11&11&11&11&11\\ 11&11&11&11&11&11&0&11&11&11&11&11\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0\\ 0&0&0&0&0&11&11&0&0&0&0&0 \end{bmatrix}$$

Note that the characteristic polynomial of $M\left(B_{5,5}^{2}\right)$ is $\lambda^{12}-2541\lambda^{10}-\lambda^{9}26620$

So, the Maximum Degree eigenvalues of $B_{5,5}^{2}$ are $0,0,\ldots\ldots,0\ (9\ \ times),-11,\ 55$ and -44.

Hence, Maximum Degree energy of $B_{n,n}^{2}=E_M\left(B_{n,n}^{2}\right)=0+|-11|+|55|+|-44|$

$=110$

Theorem 2.5: Let $n\in N, n\neq 1$. Then $E_M\left(D_2\left(B_{n,n}\right)\right)=8(n+1)\left(\sqrt{4n+1}\right)$, where $E_M\left(D_2\left(B_{n,n}\right)\right)$ is the Maximum Degree energy of graph $D_2\left(B_{n,n}\right)$.

Proof: Let $V\left(D_2\left(B_{n,n}\right)\right)=\{v_1,v_2,\ldots\ldots,v_n,v,u,u_1,u_2,\ldots\ldots u_n\}$. Note that $D_2\left(B_{n,n}\right)$ is graph with $4(n+1)$ vertices and $4(2n+1)$ edges as shown in the following Figure 5.

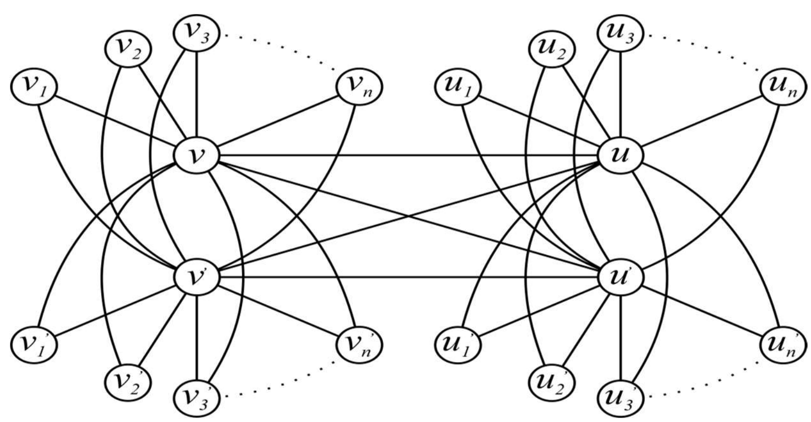

Figure 5 Shadow graph of Bistar graph $D_2\left(B_{n,n}\right)$

Observe that the Maximum Degree matrix $M\left(D_2\left(B_{n,n}\right)\right)$ of $D_2\left(B_{n,n}\right)$ is given by

$$M\left(D_2\left(B_{n,n}\right)\right) = \begin{array}{c} v_1 \\ \vdots \\ v_n \\ v_1' \\ \vdots \\ v_n' \\ v \\ v' \\ u \\ u' \\ u_1 \\ \vdots \\ u_n \\ u_1' \\ \vdots \\ u_n' \end{array} \left[\begin{array}{ccccccccccccc} v_1 & \dots & v_n & v_1' & \dots & v_n' & v & v' & u & u' & u_1 & \dots & u_n \\ 0 & \dots & 0 & 0 & \dots & 0 & 2n+2 & 2n+2 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 2n+2 & 2n+2 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 2n+2 & 2n+2 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 2n+2 & 2n+2 & 0 & 0 & 0 & \dots & 0 \\ 2n+2 & \dots & 2n+2 & 2n+2 & \dots & 2n+2 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \\ 2n+2 & \dots & 2n+2 & 2n+2 & \dots & 2n+2 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 2n+2 & 2n+2 & 2n+2 & 0 & 0 & 2n+2 & \dots & 2n+2 \\ 0 & \dots & 0 & 0 & \dots & 0 & 2n+2 & 2n+2 & 0 & 0 & 2n+2 & \dots & 2n+2 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 2n+2 & 2n+2 & 0 & \dots & 0 \end{array}\right]$$

Note that the characteristic polynomial of $M\left(D_2\left(B_{n,n}\right)\right)$ is $\lambda^{4(n+1)} - 16(2n^3 + 5n^2 + 4n + 1)\lambda^{4n+2} - 64n^2(n+1)^4\lambda^{4n}$

So, Maximum Degree eigenvalues of $M\left(D_2\left(B_{n,n}\right)\right)$ are $0,0,\dots\dots,0[4n\ times]$, $(2n+2)\left(1+\sqrt{4n+1}\right)$, $(2n+2)\left(1-\sqrt{4n+1}\right)$, $(2n+2)\left(-1+\sqrt{4n+1}\right)$ and $(2n+2)\left(-1-\sqrt{4n+1}\right)$

Hence, Maximum Degree energy of $D_2\left(B_{n,n}\right) = E_M\left(D_2\left(B_{n,n}\right)\right) = 0 + (2n+2)\left|1+\sqrt{4n+1}\right| + (2n+2)\left|1-\sqrt{4n+1}\right| + (2n+2)\left|-1+\sqrt{4n+1}\right| + (2n+2)\left|-1-\sqrt{4n+1}\right|$

$= (2n+2)\left(1+\sqrt{4n+1}-1+\sqrt{4n+1}-1+\sqrt{4n+1}+1+\sqrt{4n+1}\right)$

$= 8(n+1)\left(\sqrt{4n+1}\right)$

Example 2.6: Maximum Degree energy of shadow graph of Bistar graph $D_2\left(B_{5,5}\right) = 48\left(\sqrt{21}\right)$

Proof:

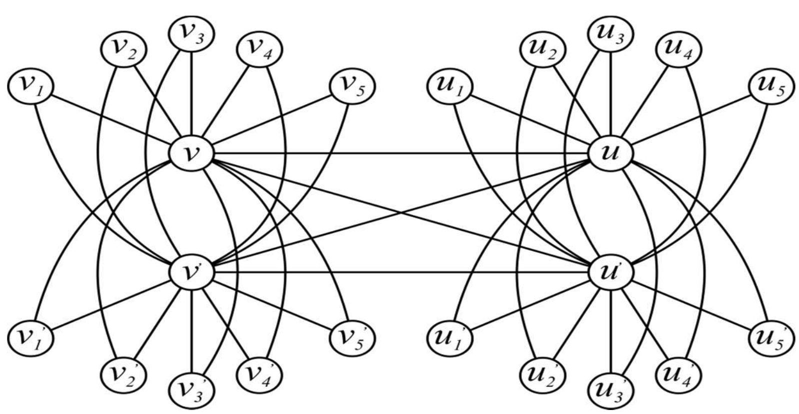

Figure 6 Shadow graph of Bistar graph $D_2(B_{5,5})$

The Maximum Degree matrix of $D_2(B_{5,5})$ is given by

$$M\left(D_2(B_{5,5})\right) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The characteristic polynomial of $M\left(D_2(B_{5,5})\right)$ is $\lambda^{24} - 6336\lambda^{22} + 8294400\lambda^{20}$

Hence, the Maximum Degree eigenvalue of $D_2(B_{5,5})$ are $0, 0, \dots\dots, 0$ (20 times), $12(1+\sqrt{21})$, $12(1-\sqrt{21})$, $12(-1+\sqrt{21})$ and $12(-1-\sqrt{21})$.

Therefore, Maximum Degree energy $E_M\left(D_2(B_{5,5})\right) = 0 + 12|1+\sqrt{21}| + 12|1-\sqrt{21}| + 12|-1+\sqrt{21}| + 12|-1-\sqrt{21}|$

$= 12(1+\sqrt{21} - 1 + \sqrt{21} - 1 + \sqrt{21} + 1 + \sqrt{21})$

$= 48(\sqrt{21})$

Theorem 2.7: Let $n \in N, n \geq 3$. Then $E_M(Gl(n)) = 2n\sqrt{2n}$, where $E_M(Gl(n))$ is the Maximum Degree energy of graph $Gl(n)$.

Proof: Let $V(Gl(n)) = \{v, v_1, v_2, \ldots \ldots, v_n, u\}$. Note that *Gl(n)* is graph with $n + 2$ vertices and $2n$ edges as shown in the following Figure 7.

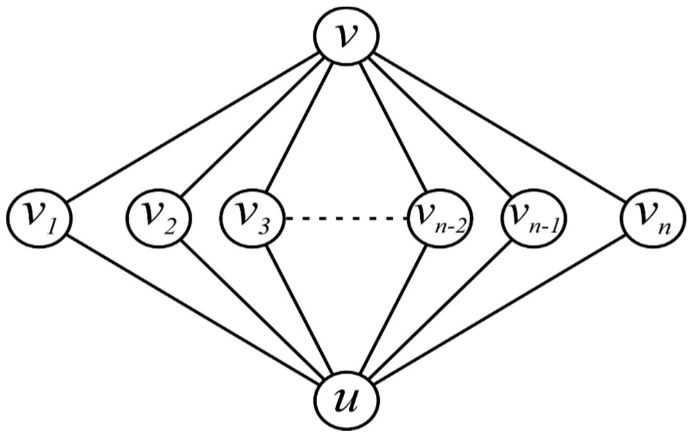

Figure 7 Globe Graph $Gl(n)$

Observe that the Maximum Degree matrix $M(Gl(n))$ of *Gl(n)* is given by

$$M(Gl(n)) = \begin{matrix} \\ v \\ v_1 \\ \vdots \\ v_n{}' \\ u \end{matrix} \begin{bmatrix} v & v_1 & \ldots & v_n & u \\ 0 & \mathrm{n} & \cdots & \mathrm{n} & 0 \\ n & 0 & \cdots & 0 & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 0 & \cdots & 0 & \mathrm{n} \\ 0 & n & \cdots & \mathrm{n} & 0 \end{bmatrix}$$

Note that the characteristic polynomial of matrix $M(Gl(n))$ is $\lambda^{n+2} - 2n^3\lambda^n$

So, eigenvalues of $M(Gl(n))$ are $0,0, \ldots \ldots ,0(n\ times), n\sqrt{2n}$ and $-n\sqrt{2n}$

Hence, Maximum Degree energy of $Gl(n) = E_M(Gl(n)) = 0 + |n\sqrt{2n}| + |-n\sqrt{2n}|$

$= n\sqrt{2n} + n\sqrt{2n} = 2n\sqrt{2n}.$

Example 2.8: Maximum Degree energy of Globe graph $Gl(5) = 10(\sqrt{10})$.

Proof:∎

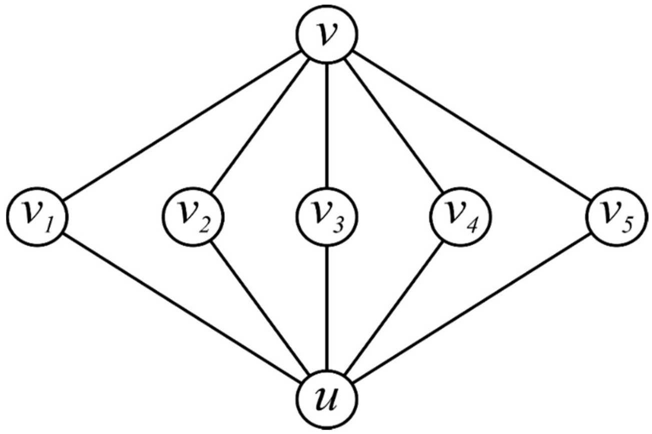

Figure 8 Globe Graph $Gl(5)$

The Maximum Degree matrix of $Gl(5)$ is given by

$$M\big(Gl(5)\big) = \begin{array}{c} \\ v \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ u \end{array}\begin{bmatrix} v & v_1 & v_2 & v_3 & v_4 & v_5 & u \\ 0 & 5 & 5 & 5 & 5 & 5 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 5 & 5 & 5 & 5 & 5 & 0 \end{bmatrix}$$

Note that the characteristic polynomial of matrix $M\big(Gl(5)\big)$ is $\lambda^7 - 250\,\lambda^5$

So, Maximum Degree eigenvalues of $Gl(5)$are $0,0, \dots ,0(5\ \ times), 5\sqrt{10},$ and $-5\sqrt{10}$.

Hence, Maximum Degree energy of $Gl(5) = 0 + \left|\, 5\sqrt{10}\right| + \left|-5\sqrt{10}\right|$.

Therefore, Maximum Degree energy of $Gl(5) = E_M\big(Gl(5)\big) = 10\sqrt{10}$

Theorem 2.9: Let $k \in N$, $k = 2n$ and $n > 2$. Then $E_M(S_k) = 4(n-1)^2$, where $E_M(S_k)$ is the Maximum Degree energy of graph S_k.

Proof: Let $V(S_k) = \{v_1, v_2, \dots \dots, v_n, u_1, u_2, \dots \dots u_n\}$. Note that S_k is graph with $2n$ vertices and $n(n-1)$ edges as shown in the following Figure 9.

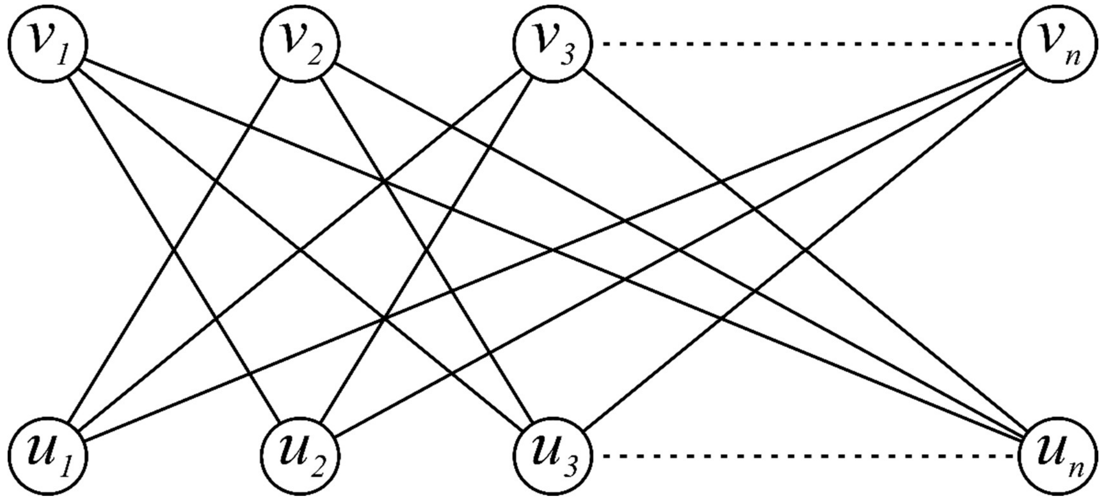

Figure 9 Crown graph S_k

Observe that the Maximum Eccentricity matrix $M(S_k)$ of S_k is given by

$$M(S_k) = \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{bmatrix} v_1 & v_2 & \cdots & v_n & u_1 & u_2 & \cdots & u_n \\ 0 & 0 & \cdots & 0 & 0 & n-1 & \cdots & n-1 \\ 0 & 0 & \cdots & 0 & n-1 & 0 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & n-1 & n-1 & \cdots & 0 \\ 0 & n-1 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ n-1 & 0 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that the characteristic polynomial of $M(S_k)$ is $(\lambda^2 - (n-1)^2)^{(n-1)}(\lambda^2 - (n-1)^4)$

So, Maximum Degree eigenvalues are $n-1, n-1, \ldots \ldots, n-1$ ($(n-1)$ $times$) $, -(n-1), -(n-1), \ldots \ldots, -(n-1)\big((n-1) times\big), (n-1)^2$ and $-(n-1)^2$.

Therefore, Maximum Degree energy of $S_k = E_M(S_k) = 0 + (n-1)|-(n-1)| + (n-1)|(n-1)| + |\,(n-1)^2| + |\,-(n-1)^2|$

$= (n-1)(n-1) + (n-1)(n-1) + (n-1)^2 + (n-1)^2$

$= 4(n-1)^2$

Example 2.10 : Maximum Degree energy of Crown graph $S_5 = 64$

Proof:

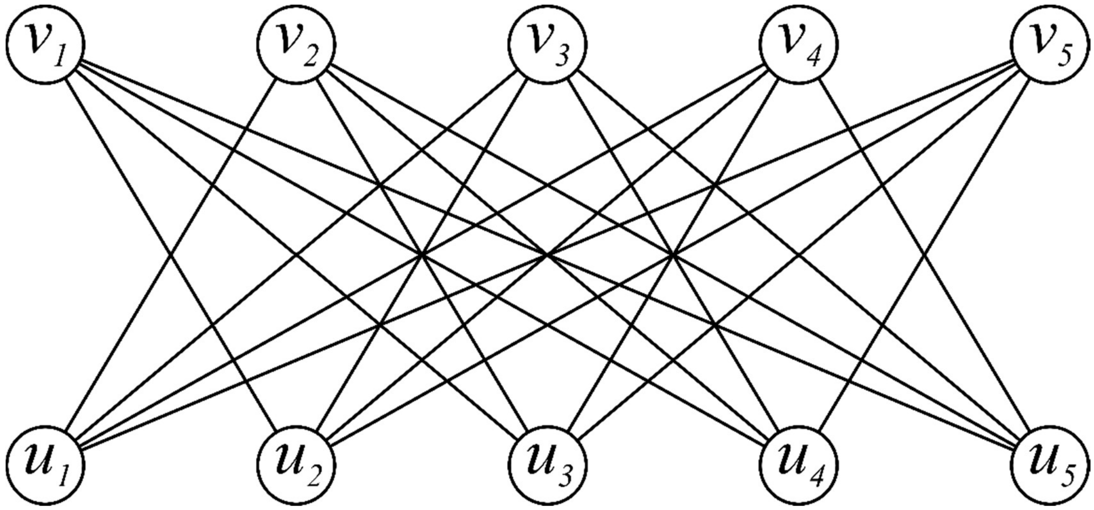

Figure 10 Crown graph S_5

The Maximum Degree matrix of S_5 is given by

$$M(S_5) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & u_1 & u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the characteristic polynomial of $M(S_5)$ is

$$\lambda^{10} - 320\lambda^8 + 17920\lambda^6 - 409600\lambda^4 + 4259840\lambda^2 - 16777216$$

So, the Maximum Degree eigenvalues of S_5 are $4,4,4,4,-4-4,-4,-4,-16$ and 16

Hence, Maximum Degree energy of $S_5 = E_M(S_5) = 4|4| + 4|4| + |-16| + |16|$

$$= 64$$

Theorem 2.11: Let $n \in N$, $n \neq 1$. Then $E_M\left(S'(K_{1,n})\right) = 2n\left(\sqrt{\frac{n}{2}(9+\sqrt{65})} + \sqrt{\frac{n}{2}(9-\sqrt{65})}\right)$, where $E_M\left(S'(K_{1,n})\right)$ is the Maximum Degree energy of graph $S'(K_{1,n})$.

Proof: Let $V\left(S'(K_{1,n})\right) = \{v_1, v_2, \dots \dots, v_n, v, v', v_1', v_2', \dots \dots v_n'\}$. Note that $S'(K_{1,n})$is graph with $2n+2$ vertices and $3n$ edges as shown in the following Figure 11.

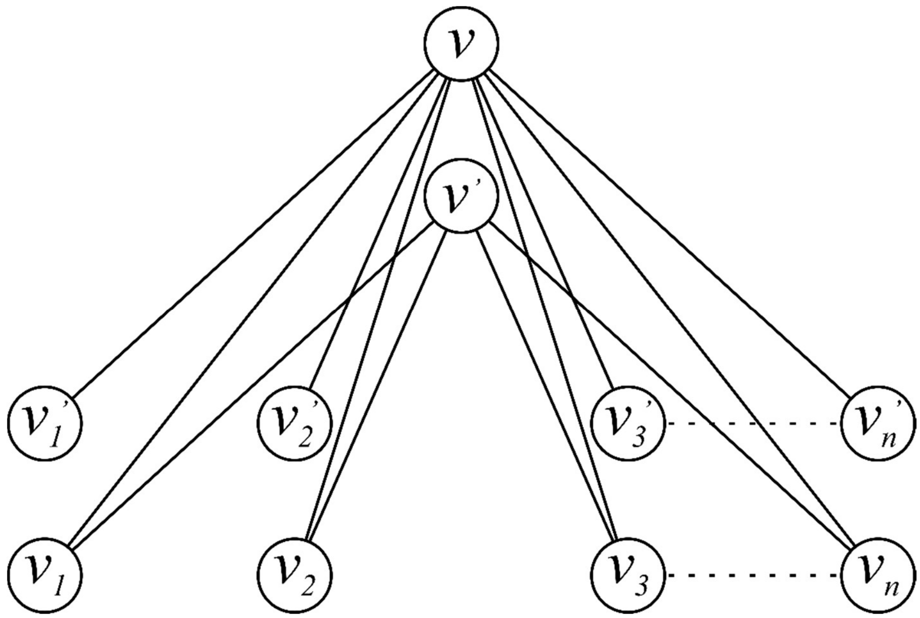

Figure 11 Splitting graph of Star graph $S'(K_{1,n})$

Observe that the Maximum Eccentricity matrix $M\left(S'(K_{1,n})\right)$ of $S'(K_{1,n})$ is given by

$$M\left(S'(K_{1,n})\right) = \begin{array}{c} \\ v_1 \\ \vdots \\ v_n \\ v \\ u \\ u_1 \\ \vdots \\ u_n \end{array} \begin{bmatrix} v_1 & \cdots & v_n & v & u & u_1 & \cdots & u_n \\ 0 & \cdots & 0 & 2n & n & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n & n & 0 & \cdots & 0 \\ 2n & \cdots & 2n & 0 & 0 & 2n & \cdots & 2n \\ n & \cdots & n & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2n & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that the characteristic polynomial of $M\left(S'(K_{1,n})\right)$ is $\lambda^{2(n+1)} - 9n^3\lambda^{2n} + 4n^6\lambda^{2(n-1)}$

So, Maximum Degree eigenvalues are $0,0, \dots \dots, 0$ $(2(n-1)\ \ times)$, $\sqrt{\frac{n^3}{2}(9+\sqrt{65})}$,

$-\sqrt{\frac{n^3}{2}(9+\sqrt{65})}$, $\sqrt{\frac{n^3}{2}(9-\sqrt{65})}$ and $-\sqrt{\frac{n^3}{2}(9-\sqrt{65})}$.

Therefore, Maximum Degree energy of $S'(K_{1,n}) = E_M\left(S'(K_{1,n})\right) = 0 + \left|\sqrt{\frac{n^3}{2}(9+\sqrt{65})}\right| + \left|-\sqrt{\frac{n^3}{2}(9+\sqrt{65})}\right| + \left|\sqrt{\frac{n^3}{2}(9-\sqrt{65})}\right| + \left|-\sqrt{\frac{n^3}{2}(9-\sqrt{65})}\right|$

$$= \sqrt{\frac{n^3}{2}(9+\sqrt{65})} + \sqrt{\frac{n^3}{2}(9+\sqrt{65})} + \sqrt{\frac{n^3}{2}(9-\sqrt{65})} + \sqrt{\frac{n^3}{2}(9-\sqrt{65})}$$

$$= 2n\left(\sqrt{\frac{n}{2}(9+\sqrt{65})} + \sqrt{\frac{n}{2}(9-\sqrt{65})}\right)$$

Example 2.12: Maximum Degree energy of Splitting graph of Star graph $S'(K_{1,5}) = 10\left(\sqrt{\frac{5}{2}(9+\sqrt{65})} + \sqrt{\frac{5}{2}(9-\sqrt{65})}\right)$

Proof:

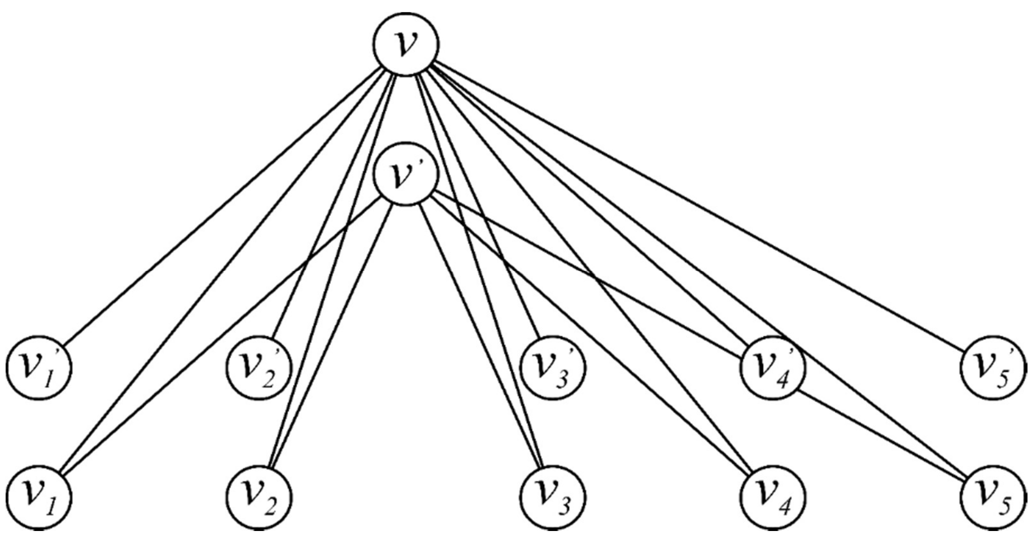

Figure 12 Splitting graph of Star graph $S'(K_{1,5})$

The Maximum Degree matrix of $S'(K_{1,5})$ is given by

$$M\left(S'(K_{1,5})\right) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v \\ u \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v & u & u_1 & u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & 0 & 0 & 0 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 10 & 10 & 10 & 10 & 10 & 0 & 0 & 10 & 10 & 10 & 10 & 10 \\ 5 & 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the characteristic polynomial of $M\left(S'(K_{1,5})\right)$ is $\lambda^{12} - 1125\lambda^{10} - 62500\lambda^{8}$

So, the Maximum Degree eigenvalues of $S'(K_{1,5})$ are $0, 0, \dots\dots, 0\ (8\ times)$, $\sqrt{\frac{5^3}{2}(9+\sqrt{65})}$, $-\sqrt{\frac{5^3}{2}(9+\sqrt{65})}$, $\sqrt{\frac{5^3}{2}(9-\sqrt{65})}$ and $-\sqrt{\frac{5^3}{2}(9-\sqrt{65})}$.

Hence, Maximum Degree energy of $S'(K_{1,5}) = E_M\left(S'(K_{1,5})\right) = 0 + \left|\sqrt{\frac{5^3}{2}(9+\sqrt{65})}\right| + \left|-\sqrt{\frac{5^3}{2}(9+\sqrt{65})}\right| + \left|-\sqrt{\frac{5^3}{2}(9-\sqrt{65})}\right| + \left|\sqrt{\frac{5^3}{2}(9-\sqrt{65})}\right|$

$$= 10\left(\sqrt{\frac{5}{2}(9+\sqrt{65})} + \sqrt{\frac{5}{2}(9-\sqrt{65})}\right)$$

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