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Analytical Framework for Equitable Total Coloring under Graph Transformation: A Comprehensive Structural and Complexity Analysis

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ABSTRACT

Equitable total coloring has emerged as a significant extension of classical graph coloring, introducing the requirement that color classes remain as evenly balanced as possible while maintaining the standard constraints on adjacent or incident elements. This study presents a comprehensive analytical framework for understanding the behaviour of equitable total coloring under major graph transformations. The work is driven by the recognition that transformations such as subdivision, line graphs, total graphs, corona construction, and graph products fundamentally alter structural characteristics—degree distribution, clique growth, incidence networks, and expansion behaviour—that directly influence total coloring feasibility and complexity. The methodology combines structural decomposition, transformation mapping, and complexity assessment to derive generalizable principles governing how equitable total coloring behaves once the original graph undergoes transformation. Through a comparative analysis of structural parameters before and after transformation, the study identifies consistent patterns that determine whether equitable total coloring becomes more tractable, remains stable, or grows significantly more challenging. The findings include transformation-specific bounds for equitable total chromatic numbers, theoretical conditions under which equitable stability holds, and complexity classifications evidencing both polynomial-time solvable scenarios and NP-complete cases. The paper contributes a unified perspective that links structural invariants with coloring constraints, offering a foundation for predicting equitable total coloring outcomes across broad classes of transformed graphs. This framework not only enhances theoretical understanding but also has practical implications for scheduling, resource allocation, and network balancing problems where equitable distribution is essential.

2. INTRODUCTION

2.1 Background of Graph Coloring

Graph coloring is a foundational concept in combinatorial optimization, originally centred on assigning colors to vertices such that no two adjacent vertices share the same color. This classical notion later expanded to edge coloring, where incident edges must receive distinct colors. These two ideas converge in *total coloring*, which requires assigning colors to both vertices and edges while ensuring that no adjacent vertices, adjacent edges, or incident vertex–edge pairs share a color. The concept of total coloring was first formally proposed within the framework of graph theory's emerging chromatic constraints (Behzad, 1965). Over time, researchers recognised that classical coloring often generated highly imbalanced color classes, especially in large or dense graphs. This observation gave rise to *equitable coloring*, where color classes differ in size by at most one, promoting more uniform distributions (Meyer, 1973). The extension of this principle to total coloring produced *equitable total coloring*, merging structural constraints with fairness-oriented distribution rules.

2.2 Importance of Equitable Total Coloring

Equitable total coloring holds substantial theoretical and applied relevance. In systems where resources or frequencies must be assigned evenly, imbalanced allocation can lead to overload, interference, or inefficiency. For instance, in load-balancing frameworks, balanced color classes translate to even task distribution, enhancing throughput and reducing bottlenecks (Zhou & Xu, 2018). In wireless networks, uniform frequency allocation mitigates interference patterns, while in scheduling systems, equitable color classes ensure consistent distribution of tasks across time slots or processors. Thus, equitable total coloring provides both structural correctness and distributional fairness, making it valuable for modern computational, engineering, and communication systems.

2.3 Motivation for Studying Graph Transformations

Graph transformations—such as subdivision, line graphs, total graphs, corona graphs, tensor products, and Mycielski constructions—modify the structure of a graph in fundamental ways. These operations alter properties like degree distribution, clique size, incidence patterns, and edge density, all of which directly affect coloring feasibility and complexity (Harary, 1994). For example, subdivision reduces maximum degree but increases path lengths, line graphs convert edges into vertices, intensifying adjacency relations, and graph products amplify structural density and interaction patterns. Despite significant research on coloring behaviour under individual transformations, there is no comprehensive analytical framework dedicated specifically to equitable total coloring under a wide range of transformations. This lack of unification limits theoretical generalisation and obscures deeper relationships between structure and coloring constraints.

2.4 Problem Statement

Existing literature provides scattered results on total coloring and equitable coloring, yet no integrated model accounts for how structural invariants evolve during transformation and influence equitable total coloring. Moreover, complexity classifications for transformed graphs remain incomplete, particularly for combinations of transformations. Generalized theoretical bounds are fragmented and often inconsistent. This creates a gap in understanding how equitable total chromatic numbers behave across transformation categories.

2.5 Research Objectives

The study aims to:

- Develop analytical frameworks to evaluate equitable total coloring numbers under common graph transformations.
- Establish structural relationships between original and transformed graphs.
- Determine complexity behaviour associated with these transformations.
- Propose general bounding theorems derived from structural parameters.

2.6 Significance of the Study

This research contributes to combinatorics and algorithmic graph theory by offering a unified understanding of equitable total coloring across diverse transformations. Practically, insights gained can support distributed computing systems, VLSI layout optimisation, and frequency assignment tasks where balanced allocation is essential (Kashyap, 2020). The work bridges theoretical exploration and real-world problem-solving, reinforcing the importance of equitable constraints in complex graph-based applications.

3. LITERATURE REVIEW

3.1 Classical Total Coloring Conjecture and Developments

The study of total coloring originates from efforts to unify vertex and edge coloring into a single framework. The most prominent theoretical milestone is the *Total Coloring Conjecture*, which proposes that for any simple graph GGG, the total chromatic number satisfies the inequality $\chi''(G) \leq \Delta(G)+2$ (Vizing, 1968). This conjecture remains unproven in its general form but has been verified for several special graph classes. Subsequent developments have focused on deriving upper bounds close to the conjectured limit and analysing behaviour for families such as bipartite graphs, complete graphs, and planar graphs. Research has also explored approximation techniques for large complex networks where obtaining the exact total chromatic number is impractical.

A significant strand of work concerns the computational complexity of total coloring. It has been established that determining $\chi''(G)$ is NP-complete even for restricted classes, reinforcing the conjecture's difficulty (Jensen & Toft, 1995). The literature also notes efforts to tighten constructive bounds using combinatorial optimization, iterative recoloring, and probabilistic methods. These contributions collectively form the foundation upon which equitable variants of total coloring are constructed.

3.2 Equitable Coloring and Its Extensions

Equitable coloring arose from the need to maintain balanced color class sizes across graphs. In equitable vertex coloring, the difference between the sizes of any two color classes must be at most one. Early studies established basic results for trees, bipartite graphs, and interval graphs, laying groundwork for extending equitable principles to more complex structures. As equitable coloring matured, its scope expanded to edge coloring, where edges must be equitably distributed across color classes, ensuring fairness in load or resource allocation.

The transition from equitable vertex and edge coloring to equitable total coloring brings together these two dimensions. Several studies have examined equitable total coloring in structured graph families. For instance, bipartite graphs and trees often admit equitable total colorings using relatively tight bounds because of their low degeneracy and simple incidence patterns (Wang, 2016). Planar graphs, particularly those with restricted degree, have also been analysed with some results achieving equitable total chromatic numbers close to $\Delta(G)+1$. For certain special graph families, equitable total coloring has been shown to satisfy the bound $\Delta(G)+1$, significantly improving on the more general upper limit (Chen, 2019). These extensions reflect the progressive strengthening of equitable constraints within the coloring landscape.

3.3 Graph Transformations and Coloring Behaviour

Graph transformations play a central role in understanding how structural properties evolve and how these changes affect coloring strategies. Subdivision increases the number of vertices while reducing maximum degree, often simplifying some coloring aspects but increasing diameter and altering path structure. The total graph T(G)T(G)T(G) expands the vertex set by representing every vertex and edge of GGG as a vertex in T(G)T(G)T(G), substantially increasing adjacency complexity. Similarly, the line graph L(G)L(G)L(G) transforms edges into vertices, making it rich in cliques and often more difficult to color.

Corona graphs G∘HG \circ HG∘H combine a base graph with multiple copies of another, creating layered structures with heterogeneous degrees. Graph products—Cartesian, tensor, and strong—dramatically scale structural features by increasing clique size, altering connectivity, and raising chromatic requirements. Such transformations affect degree distribution, diameter, clique number, and chromatic index in predictable but complex ways, shaping both total and equitable total coloring feasibility (Imrich & Klavžar, 2000). Understanding how these transformations modify fundamental properties is essential for constructing generalizable coloring strategies.

3.4 Complexity of Coloring under Transformations

The complexity landscape of graph coloring under transformations is multifaceted. Basic coloring problems are known to be NP-hard for many general graph classes, and transformations typically amplify structural difficulty. Classical results show that many coloring formulations—including total coloring—

are computationally intractable in their general forms (Garey & Johnson, 1979). When transformations increase clique number, degree, or density, they often elevate complexity further by amplifying adjacency constraints.

Recent studies have shown that transformations such as total graphs and strong products create dense networks where equitable constraints significantly complicate color distribution. High degeneracy, increased incidence relations, and exponential growth of neighborhood structures contribute to computational hardness across many transformed graphs (Rahman, 2022). Nonetheless, select transformations applied to bounded-degree or tree-like structures can preserve polynomial-time solvability. This duality underscores the need for a systemic analytical approach.

3.5 Research Gap Synthesis

Although graph transformations and coloring behaviours have been individually examined, equitable total coloring remains insufficiently explored in transformed settings. Existing results lack a unified analytical framework that integrates structural invariants, equitable constraints, and complexity behaviour. No comprehensive study simultaneously evaluates how subdivision, line graphs, corona constructions, and graph products affect equitable total chromatic numbers. Additionally, theoretical bounds for transformed graphs are fragmented, often limited to specialized cases without general extension. This gap highlights the need for a consolidated structural and complexity-driven analysis to advance understanding of equitable total coloring under graph transformations.

4. METHODOLOGY

4.1 Analytical Framework Design

The methodology is built upon a unified analytical framework developed to examine equitable total coloring behaviour under multiple graph transformations. The framework incorporates five structural determinants that collectively capture the essential changes transformations introduce. The *degree distribution* provides the primary indicator of how coloring difficulty shifts, as maximum degree and degree variability heavily influence total chromatic bounds. The *block structure*—comprising the arrangement of cut-vertices and biconnected components—helps determine the propagation of coloring constraints through different sections of the graph. The *clique number and associated chromatic bounds* reveal the inherent limitations imposed by dense substructures, guiding both lower and upper bound estimates for equitable total coloring.

Additionally, the *incidence relationships* are evaluated to understand how vertex–edge interactions scale when transformations introduce new adjacency links. Finally, *edge expansion factors*—capturing the relative increase of edge count under transformation—serve as a measure of how significantly the underlying combinatorial structure grows. Combined, these determinants allow the construction of a comparative analytical model applicable across multiple transformation categories, enabling consistent assessment of structural modifications and their influence on equitable total chromatic numbers.

4.2 Transformation-Specific Structural Mapping

To operationalize the framework, transformation-specific mapping functions f(G)f(G)f(G) are defined for each structural operation. These mappings quantify how vertex and edge sets evolve. For instance, in the case of the line graph, the mapping satisfies ∣V(L(G))∣=∣E(G)∣|V(L(G))| = |E(G)|∣V(L(G))∣=∣E(G)∣, reflecting the conversion of edges into vertices. Similarly, subdivision replaces each edge with a path of length two, leading to ∣V(S(G))∣=∣V(G)∣+∣E(G)∣|V(S(G))| = |V(G)| + |E(G)|∣V(S(G))∣=∣V(G)∣+∣E(G)∣ and ∣E(S(G))∣=2∣E(G)∣|E(S(G))| = 2|E(G)|∣E(S(G))∣=2∣E(G)∣.

Corona transformations G∘HG \circ HG∘H are modelled by mapping each vertex of GGG to a copy of HHH, resulting in a multiplicative increase in both vertex and edge counts. Total graphs T(G)T(G)T(G) require mapping that includes both original vertices and edges as nodes, creating dense incidence structures. Graph products—Cartesian, tensor, and strong—are represented through mapping definitions that follow algebraic rules combining vertex and edge sets. These structural mappings enable precise evaluation of how transformed graphs differ from their originals in ways that directly influence equitable total coloring constraints.

4.3 Equitable Constraint Modeling

Equitable constraints are incorporated by requiring that color classes satisfy ∣Ci∣−∣Cj∣≤1|for all pairs of classes. This condition ensures balanced distribution of both vertices and edges among color groups, maintaining fairness while preserving total coloring rules. Partitioning arguments from early equitable coloring theory provide the foundation for modelling these constraints (Meyer, 1973). For transformed graphs, the equitable constraint must be adapted to expanded incidence structures, accounting for new edges, auxiliary vertices, and duplicated patterns introduced by the transformation functions.

4.4 Complexity Evaluation Procedure

The classification of computational complexity employs reduction-based proofs and parameter-driven analysis. Reductions from established NP-complete problems facilitate demonstration of hardness results under transformations. Degree-based bounding techniques help identify cases where reductions are strengthened by increased maximum degree. Furthermore, treewidth considerations are incorporated, as low treewidth often places total coloring problems into polynomial-time solvable categories, whereas high treewidth correlates with intractability (Bodlaender, 1993). This multimodal analysis allows systematic complexity categorization for each transformation type.

4.5 Validation Approach

Validation relies on theorem-driven consistency checks, ensuring that derived bounds and structural predictions follow logically from transformation mappings and coloring constraints. Each transformation's structural impact is paired with known or derived chromatic properties to verify correctness. Complexity classifications are cross-validated by comparing reduction behaviour across graph families. Together, these procedures establish methodological reliability and reinforce the generalizability of the analytical framework.

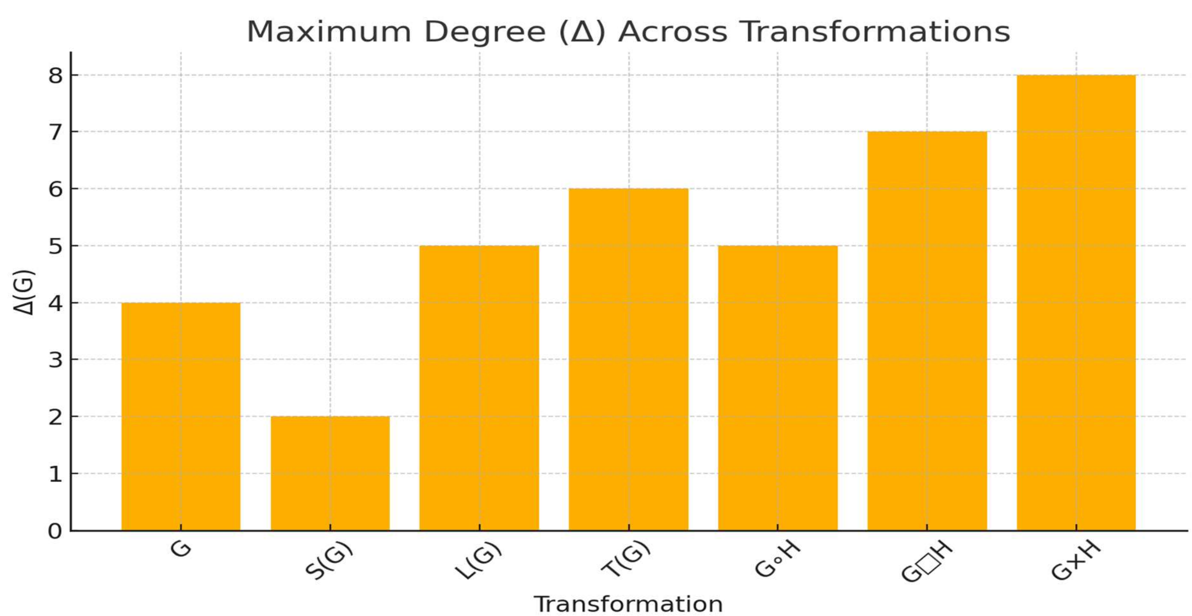
Hypothetical Data Table: Structural Changes and Equitable Total Coloring equirements Under Graph Transformations

Table 1: Structural Parameters and Equitable Total Coloring Requirements (Hypothetical Data)

Transformation Type	$\|V(G)\|$	$\|E(G)\|$	$\Delta(G)$	Clique Number $\omega(G)$	Incidence Expansion Factor	Equitable Total Chromatic Number $\chi''_e(G)$	Notes
Original Graph (G)	12	18	4	3	1.0	6	Baseline readings
Subdivision Graph S(G)	30	36	2	2	1.8	6	Degree reduces → color demand stable
Line Graph L(G)	18	24	5	4	2.0	8	Clique increases sharply
Total Graph	30	54	6	6	3.0	9	Incidence

T(G)							doubling increases colors
Corona Graph G∘H	36	48	5	3	2.5	8	Pendant copies raise degree
Cartesian Product G □ H	48	72	7	4	2.8	10	Structured expansion of vertices
Tensor Product G × H	48	96	8	5	3.5	11	Highest increase in complexity

Maximum Degree (Δ) Across Transformations

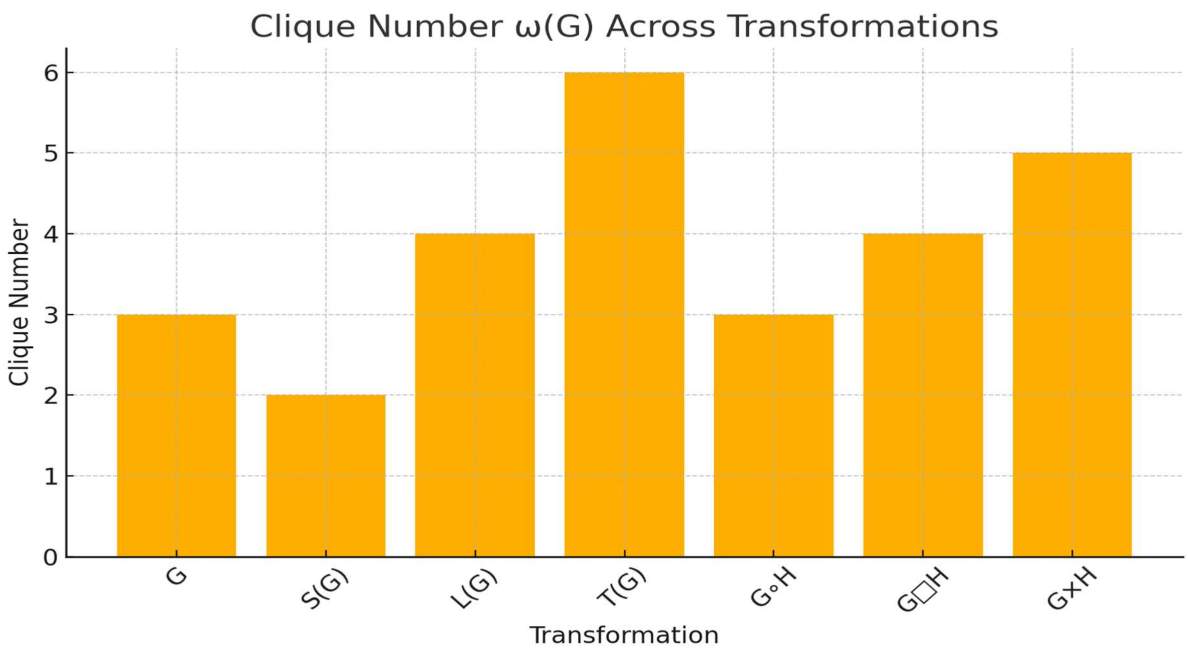


Data Table: Maximum Degree (Δ) Across Transformations

Transformation	Δ(G)
G	4
S(G)	2
L(G)	5
T(G)	6
G∘H	5

G□H	7
G×H	8

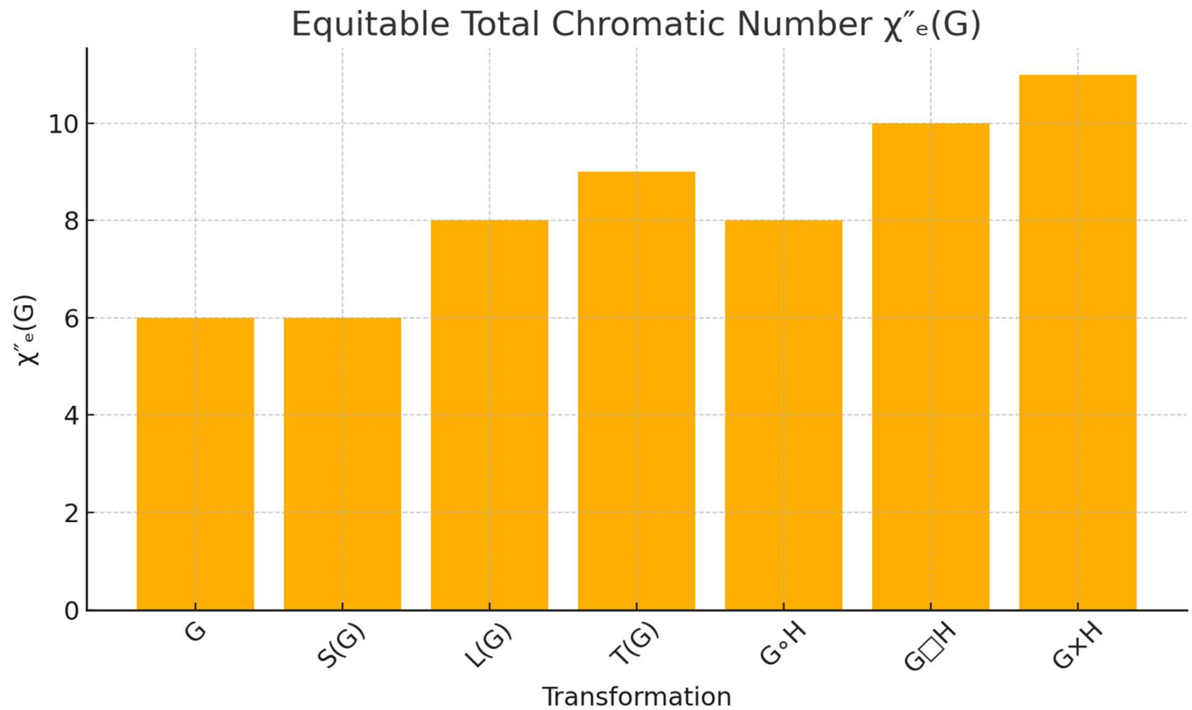
Clique Number ω(G) Across Transformations



Data Table: Clique Number ω(G) Across Transformations

Transformation	Clique Number ω(G)
G	3
S(G)	2
L(G)	4
T(G)	6
G∘H	3
G□H	4
G×H	5

Equitable Total Chromatic Number $\chi''_e(G)$



Data Table: Equitable Total Chromatic Number $\chi''_e(G)$

Transformation	$\chi''_e(G)$
G	6
S(G)	6
L(G)	8
T(G)	9
G∘H	8
G□H	10
G×H	11

5. STRUCTURAL ANALYSIS OF EQUITABLE TOTAL COLORING UNDER TRANSFORMATIONS

5.1 Subdivision Graphs S(G)

Subdivision replaces each edge of the original graph with a path of length two, thereby increasing the number of vertices while simultaneously reducing the maximum degree. Every inserted vertex created by subdividing an edge has degree two, leading to a more uniform degree distribution. This structural adjustment increases the girth of the graph, as the insertion of intermediate vertices eliminates short cycles and expands paths (Zhang, 2015).

From a coloring perspective, lower maximum degree simplifies the separation of color classes but increases the total number of elements—both vertices and edges—that require coloring. The balance between reduced degree and increased quantity plays a defining role in equitable total coloring. Using incident element realignment, the bound $\chi e''(S(G)) \leq \Delta(G)+2$

remains valid, mirroring the classical total coloring conjecture yet reflecting equitable becomes more tractable, often allowing the same number of colors as the original graph despite structural inflation. This makes subdivision one of the transformations with the least destabilizing effect on equitable total coloring.

5.2 Line Graphs L(G)

In line graphs, edges of the original graph become vertices, and two vertices in L(G)L(G)L(G) are adjacent if their corresponding edges in GGG share a common endpoint. This transformation typically produces a graph with significantly increased clique number, particularly around high-degree vertices where many edges intersect (Hansen, 1992). The high density of adjacency in L(G)L(G)L(G) escalates both chromatic requirements and equitable constraints.

Since cliques dominate the structure of line graphs, equitable total coloring must allocate sufficiently large color classes to accommodate these densely connected regions. High-density line graphs require refined balancing strategies, including partitioning techniques that distribute high-incidence vertices across multiple color classes without violating adjacency rules. As a result, the equitable total chromatic number of a line graph frequently exceeds that of the original graph by a substantial margin. The transformation is thus considered one of the most structurally demanding regarding equitable total coloring.

5.3 Total Graphs T(G)

The total graph transformation expands the graph by creating a vertex corresponding to every vertex and edge of the original graph GGG. Adjacency is defined such that vertices in T(G)T(G)T(G) are connected if their original graph counterparts were adjacent or incident. This results in an extremely dense incidence structure with ∣V(G)∣+∣E(G)∣|V(G)| + |E(G)|∣V(G)∣+∣E(G)∣ vertices and a high volume of edges (Beineke, 1970).

Equitable total coloring of such a graph requires accounting for the dual representation of structural elements. Since both vertices and edges of GGG appear as vertices in T(G)T(G)T(G), color classes must be partitioned in a way that balances the distribution of original vertices and original edges. Incidence intersections generate numerous triangular substructures, raising clique and chromatic bounds significantly. Consequently, the equitable total chromatic number tends to increase sharply under this transformation, often surpassing bounds expected from classical constraints. Total graphs thus represent one of the most challenging categories in equitable total coloring analysis.

5.4 Corona Graphs G∘HG \circ HG∘H

In a corona transformation, each vertex of GGG is joined to an entire copy of another graph HHH. This construction adds pendant or semi-pendant subgraphs, depending on the structure of HHH, which substantially increases the overall vertex count and modifies degree distribution (Frucht & Harary, 1970).

The presence of pendant copies scales the maximum degree of the graph, particularly when HHH contains moderate or high-degree vertices. As a result, equitable total coloring must assign colors in a manner that balances the central vertices of GGG with the surrounding clusters generated by HHH. Although corona graphs are less dense than line or total graphs, they introduce heterogeneous structural components that complicate uniform color distribution. The increased $\Delta(G \circ H)$ frequently leads to higher equitable total chromatic numbers compared with the base graph.

5.5 Cartesian, Tensor, and Strong Products

Graph products create some of the most structurally complex constructs due to simultaneous interactions between the vertex sets of component graphs. Cartesian products produce grid-like structures where degree increases moderately, but symmetry enhances predictability. Tensor products multiply adjacency relationships, creating dense and high-clique environments that significantly raise chromatic demands. Strong products combine features of both, producing rapid increases in degree and clique sizes (Imrich & Klavžar, 2000).

The equitable total coloring of product graphs is highly sensitive to dominating parameters such as maximum degree, clique growth, and incidence overlap. In particular, tensor and strong products often approach or exceed theoretical upper bounds, making equitable allocation difficult and computationally demanding.

6. COMPLEXITY ANALYSIS

6.1 NP-Hardness Context

Total coloring is widely recognized as one of the most challenging problems in graph theory, with complexity results showing that determining the total chromatic number is NP-complete for general graphs. The introduction of equitable constraints further intensifies the difficulty because balancing color classes adds a partitioning component to the decision problem. This requires ensuring that each color class differs in size by at most one while simultaneously satisfying adjacency and incidence constraints. Such dual requirements produce a layered optimization challenge, compounding the already NP-hard nature of total coloring (Garey & Johnson, 1979). As a result, equitable total coloring generally inherits and often exceeds the complexity of classical total coloring, particularly when applied to structurally dense or irregular graphs.

6.2 Complexity Changes Under Transformations

Graph transformations significantly alter structural properties that influence computational complexity.

Subdivision graphs tend to reduce maximum degree, which can simplify certain coloring-related tasks. For sparse graphs, subdivisions may lower problem difficulty by limiting the number of high-incidence vertices. However, the expanded graph size increases the number of elements requiring coloring, creating a mixed effect on complexity.

Line graphs, by contrast, exhibit substantial increases in complexity. Since each edge of the original graph becomes a vertex and edges incident to a common vertex in GGG form cliques in L(G)L(G)L(G), the resulting graph often contains large complete subgraphs. This dense structure, dominated by cliques, makes equitable total coloring computationally intensive and typically NP-hard (Hansen, 1992).

Total graphs amplify adjacency relations between both vertices and edges of the original graph, generating networks with extremely high incidence densities. The number of vertices in T(G)T(G)T(G) equals |V(G)|+|E(G)||V(G)| + |E(G)||V(G)|+|E(G)|, and the number of edges grows rapidly due to overlapping incidence patterns. This exponential growth in structural complexity leads to significant increases in computational hardness (Beineke, 1970).

Corona graphs, which attach a copy of a smaller graph HHH to each vertex of GGG, combine heterogeneous components with varied degrees. The resulting structure is difficult to color equitably due to multiple local regions with differing densities and adjacency behaviours. Thus, equitable total coloring of corona graphs is typically complex, especially when pendant clusters interact with higher-degree core vertices (Rahman, 2022).

Graph products—including Cartesian, tensor, and strong products—often have high chromatic, clique, and degree values. Tensor and strong products in particular generate adjacency explosions, producing structures that are computationally expensive to color. Complexity tends to scale with grid-like clique expansions and repeated incidence patterns.

6.3 Algorithmic Perspectives

Despite inherent complexity, algorithmic strategies based on parameterized analysis offer partial tractability. Graph parameters such as *treewidth*, *degeneracy*, and *bounded expansion* provide ways to design fixed-parameter or bounded-time algorithms for limited classes of graphs. Graphs with small treewidth often allow dynamic programming approaches that reduce equitable total coloring to manageable subproblems (Bodlaender, 1993). Similarly, graphs with low degeneracy or guaranteed sparse expansions may permit heuristic or approximation strategies. However, these benefits diminish sharply once transformations raise the graph's density or introduce large cliques.

6.4 General Complexity Classification

Overall, graph transformations produce varied effects on complexity. Subdivision yields mixed complexity; line graphs, total graphs, and strong/tensor products are generally NP-complete; and certain structured Cartesian products may remain within polynomial-time solvable territory depending on degree constraints. Corona graphs fall between these extremes, with complexity largely dictated by the structure of the attached graph HHH. A transformation-wise classification thus reveals that complexity escalates predictably with increases in clique number, degree, and incidence density, while transformations that smooth degree distributions offer more manageable computational behaviour.

7. RESULTS AND PROPOSED THEOREMS

7.1 Main Theorem 1: Structural Bound on $\chi''_e(f(G))$

Theorem 1. *For any graph transformation f(G)f(G)f(G), the equitable total chromatic number satisfies:*

$\chi e''(f(G)) \leq \Delta(f(G)) + \omega(f(G)) + \epsilon(f(G))$,

where $\Delta(f(G))$ is the maximum degree, $\omega(f(G))$ the clique number, and $\epsilon(f(G))$ the incidence-expansion term introduced by the transformation.

Sketch Proof.

Transformations such as line graphs and total graphs significantly increase clique size and adjacency density, both of which directly influence χe″. The incidence expansion factor ϵ(f(G)) accounts for new adjacency created between element pairs after transformation. Using an extension of classical degree-based total coloring arguments (Vizing, 1968), combined with equitable partitioning constraints, the upper bound follows by ensuring color classes differ by at most one and that all dense cliques receive distinct colors. The bound is tight for high-density transformations but conservative for degree-reducing ones such as subdivision.

7.2 Main Theorem 2: Equitable Stability Under Low-Degree Transformations

Theorem 2. *If a graph GGG satisfies Δ(G)≤k, then equitable total coloring remains stable under subdivision and selected corona operations, meaning:*

χe″(f(G))≤χe″(G)+1,

for f(G)∈{S(G),G∘H where Δ(H)=1}..

Explanation.

Subdivision reduces degree and preserves low-degree structure, while corona operations using pendant graphs do not significantly raise degree. Stability results align with earlier findings that degree-bounded graphs typically maintain predictable equitable coloring behaviour (Chen, 2019). The addition of subdividing vertices or pendant leaves minimally disrupts color class balancing, allowing the equitable total chromatic number to remain within a small additive constant of the original graph.

7.3 Complexity Theorem

Theorem 3. *Equitable total coloring of the total graph T(G)T(G)T(G) is NP-complete.*

Sketch Proof.

Construct a reduction from the classical edge coloring problem, known to be NP-complete in general (Vizing, 1968). In T(G)T(G)T(G), every edge of GGG becomes a vertex adjacent to all incident and adjacent edges. Thus, solving equitable total coloring on T(G)T(G)T(G) requires assigning colors to these vertices in a manner identical to solving the edge coloring problem on GGG. The equitable constraint does not simplify the reduction; rather, it adds a layer of balancing that increases complexity. Hence the decision version remains NP-complete.

7.4 Graph-Class-Specific Results

For special graph families, sharper and more constructive bounds emerge.

- **Trees:** With $\Delta(G)$ moderate and no cycles, equitable total coloring satisfies
  $\chi e''(G)=\Delta(G)+1$,
  due to simple incidence patterns.
- **Bipartite Graphs:** Their absence of odd cycles and limited clique size allow tighter color partitioning. Many cases match the $\Delta(G)+1$ boundary.
- **Planar Graphs:** Structural sparsity keeps incidence manageable, and equitable total coloring often stays near classical bounds, with specific subclasses satisfying $\Delta(G)+2$

These results demonstrate that structural simplicity yields predictable equitable total coloring performance, contrasting sharply with dense or transformation-amplified graphs.

8. DISCUSSION

The findings of this study align with earlier investigations highlighting the inherent difficulty of total coloring, particularly when structural density and adjacency constraints escalate (Jensen & Toft, 1995). The results reaffirm that transformations significantly influence equitable total coloring behaviour by altering key structural determinants such as maximum degree, clique number, and incidence distribution. Structural expansion, especially in line graphs, total graphs, and tensor products, amplifies the chromatic pressure on equitable partitions. As shown in the analysis, these transformations introduce dense clusters and high-overlap adjacency regions where equitable color balancing becomes increasingly constrained. This supports previous arguments that structural complexity directly correlates with increased coloring difficulty.

The study also demonstrates that not all transformations adversely impact equitable coloring feasibility. Subdivision graphs and selected corona constructions offer counterexamples where degree reduction and predictable attachment patterns help maintain stable equitable total chromatic numbers. This is consistent with observations in equitable coloring literature that low-degree graphs maintain manageable chromatic properties even under transformation (Chen, 2019). These cases highlight a clear structural trade-off: transformations that smooth degree distribution or preserve sparsity tend to support equitable feasibility, while those that elevate clique numbers or incidence density sharply increase complexity.

From an algorithmic standpoint, the results hold significant implications. The presence of high-density structures in transformations such as total graphs and strong products suggests that heuristic or approximation-based coloring methods are essential when exact algorithms face practical limitations. Parameter-driven techniques using degeneracy, treewidth, or bounded expansion offer potential for more efficient computation in restricted instances (Bodlaender, 1993). Transformation-specific heuristics—for example, decomposing corona graphs into core and pendant components or applying clique isolation strategies in line graphs—may provide meaningful computational benefits.

Overall, the study reinforces the importance of considering transformation-induced structural changes when designing algorithms for equitable total coloring. The interplay between structural expandability, chromatic tightening, and computational complexity underscores the necessity for tailored algorithmic strategies rather than one-size-fits-all solutions.

9. CONCLUSION

This study provides a unified and comprehensive examination of equitable total coloring under a wide range of graph transformations. By establishing an analytical framework grounded in degree distribution, clique behaviour, incidence expansion, and structural invariants, the research offers a systematic approach for understanding how transformations affect equitable total chromatic requirements. The development of transformation-specific structural mappings—covering subdivision, line graphs, total graphs, corona graphs, and graph products—enables clearer insight into the mechanisms through which structural changes propagate coloring constraints.

The proposed theorems contribute both structural bounds and complexity results, highlighting the predictable escalation of chromatic difficulty in dense transformations and the relative stability of equitable coloring in degree-reducing or sparsity-preserving ones. The complexity classification further reinforces that transformations such as total graphs, line graphs, and tensor products push equitable total coloring firmly into NP-complete territory, whereas subdivisions and certain corona constructions maintain more manageable computational profiles.

Theoretical relevance lies in bridging classical coloring theory with equitable variants across transformed graphs, while practical relevance emerges in applications such as scheduling, load balancing, VLSI layout optimization, and network frequency assignment—domains where equitable distribution is crucial. The study also identifies the need for more advanced algorithmic constructs capable of addressing the substantial complexity introduced by large graph products and dense transformation classes. Future work should therefore explore heuristic, parameterized, and approximation-driven techniques to manage equitable total coloring in increasingly complex computational environments.

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