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Computing SK_1 Index of k-Splitting Graph of Generalized Transformation Graphs

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ABSTRACT

Recently, in [18] Raikar et al. obtained explicit formulae for the first Zagreb index and second zagreb index of k-splitting of generalized transformation graphs and their complement. In this paper we obtain explicit formulae for the SK_1 index of k-splitting of generalized transformation graphs $spl_k(G^{ab})$

2010 Mathematics Subject Classifications: 05C05, 05C07, 05C35.

1 Introduction

The vertices of molecular graph correspond to the atoms in the molecule and the edges correspond to the covalent bonds. A topological index is a graph invariant which maps each molecular graph to a real number. Numerous such descriptors have been considered in theoretical chemistry, and have found some applications, especially in QSPR/QSAR [14, 21]. Topological indices are categorized into two types namely degree-based indices and distance-based indices. Some of the well known degree-based indices are first Zagreb index, second Zagreb index, forgotten index, hyper Zagreb index, Randic index, harmonic index, geometric-arithmatic index, redefined third Zagreb index, etc. For more details on degree based topological indices we refer to [6, 7, 8, 9, 10].

Among all these degree-based indices the first and second Zagreb indices are studied widely. In 1972, Gutman et al. [7] defined the first and second Zagreb indices of a graph G. Till today, many researcheRs over the globe are exploring these indices to the advanced levels. Later in 2008, Došlic$'$ [5] defined the first and second Zagreb coindices, due to all non-adjacent pair of vertices. For more information on the Zagreb indices and coindices with their applications see [2, 10, 15, 16, 17, 20, 24]. In 2015, Basavanagoud et al. [3] defined new graph operations called generalized transformation G^{ab}, and obtained the expressions for the first and second Zagreb indices and coindices of these graphs and their complements. In 2017, Vaidya et al. [23] defined a new graph operation called k-splitting of a graph G and studied energy of the same.

Recently, in [18] Raikar et al. obtained explicit formulae for the first Zagreb index and coindex of k-splitting of generalized transformation graphs and their complement. Motivated by this, in the present work we focus on obtaining explicit formulae for the second Zagreb index and coindex of k-splitting of generalized transformation graphs $spl_k(G^{ab})$, and followed by this we also obtain analogous expressions for the complements of $spl_k(G^{ab})$.

2 Preliminaries

Throughout this paper, we consider undirected, simple and finite graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$, where members of $V(G)$ are called vertices and members of $E(G)$ are called edges of graph G. Two vertices are said to be adjacent in G if they share a common edge and this edge is said to be incident on those two vertices. The degree of a vertex u in $V(G)$, denoted by $d_G(u)$, is the number of edges which are incident with u. The complement $\overline{G}$ of the graph G is a simple graph on the same vertex set of G such that two vertices u and v are adjacent in G if and only if they are non-adjacent in $\overline{G}$. For other graph theoretic definitions, we refer the book [13].

The first Zagreb index $M_1(G)$ [7] and second Zagreb index $M_2(G)$ [7] are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \quad and \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)\ d_G(v)$$

respectively.

The first Zagreb can also be expressed as [4]

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

The first Zagreb coindex $\overline{M_1}(G)$ [5] and second Zagreb coindex $\overline{M_2}(G)$ [5] are difined as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)] \quad and \quad \overline{M_2}(G) = \sum_{uv\notin E(G)} d_G(u)\ d_G(v)$$

SK,SK$_1$,SK$_2$ indices [24] of a graph are defined as

$$S\,K(G)=\sum_{uv\in E(G)}\left\lceil\frac{d_G(u)+d_G(v)}{2}\right\rceil.$$

$$SK_1(G)=\sum_{uv\in E(G)}\left\lceil\frac{d_G(u)d_G(v)}{2}\right\rceil.$$

$$SK_1(G)=\sum_{uv\in E(G)}\frac{[d_G(u)+d_G(v)]^2}{4}$$

respectively.

Let G be a graph with vertx set $V(G)$ and edge set $E(G)$, and let α,β be two elements of $V(G)\cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise is $-$. Let ab be a 2-permutation of the set $\{+,-\}$. We say that α and β correspond to the first term a of ab if both α and β are in $V(G)$, whereas α and β correspond to the second term b of ab if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{ab} of G is defined on the vertex set $V(G)\cup E(G)$. Two vertices α and β of G^{ab} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of ab. With respect to this, there are four graphical transformations of graphs such as G^{++},G^{+-},G^{-+},G^{--}, as there are four distinct 2-permutations of $\{+,-\}$.

In another way, the generalized transformation graph G^{ab} is a graph whose vertex set is $V(G)\cup E(G)$, and $\alpha,\beta\in V(G^{ab})$ such that α and β are adjacent in G^{ab} if and only if (i) and (ii) holds:

(i) $\alpha,\beta\in V(G^{ab})$, α,β are adjacent in G if $a=+$ and α,β are not adjacent in G if $a=-$.

(ii) $\alpha\in V(G)$ and $\beta\in E(G)$, α, β are incident in G if $b=+$ and α,β are not incident in G if $b=-$.

The vertex u of G^{ab} corresponding to a vertex u of G is referred to as a point vertex. The vertex e of G^{ab} corresponding to an edge e of G is referred to as a line vertex.

The k-splitting of a graph G, denoted by $spl_k(G)$, is obatined by adding to each vertex u of G new k vertices, say u', u'', ..., $u^{(k)}$ such that $u^{(i)}$, $1\le i\le k$, is adjacent to each vertex that is adjacent to u in G.

The following results are required for the present considerations.

Lemma 2.1 *[1, 10] For a graph G with n vertices and m edges*

$$M_1(\overline{G})=M_1(G)+n(n-1)^2-4m(n-1).$$

Lemma 2.2 *[10, 20] For a graph G with* n *vertices and* m *edges*

$$\overline{M_1}(G) = 2m(n-1) - M_1(G).$$

Lemma 2.3 *[10, 20] For a graph G with* n *vertices and* m *edges*

$$\overline{M_1}(\overline{G}) = \overline{M_1}(G).$$

Lemma 2.4 *[10, 20] For a graph G with* n *vertices and* m *edges*

$$M_2(\overline{G}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(G) - M_2(G).$$

Lemma 2.5 *[10, 20] For a graph G with* n *vertices and* m *edges*

$$\overline{M_2}(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G).$$

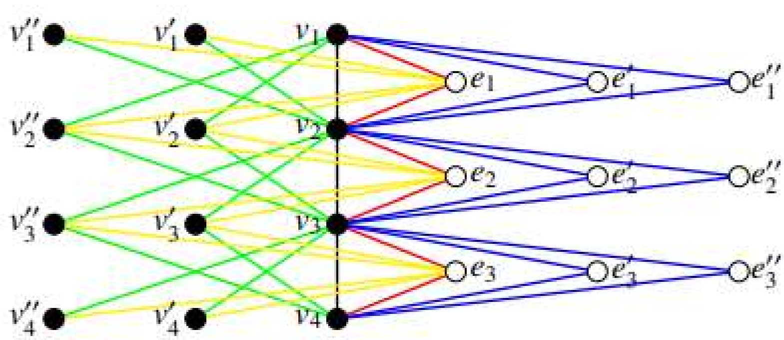
Lemma 2.6 *[10, 20] For a graph G with* n *vertices and* m *edges*

$$\overline{M_2}(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G).$$

Now we present the main results of our work through following sections.

3 SK1 Index of $\boldsymbol{spl_k(G^{++})}$

Let G be a (n,m)-graph and $spl_k(G^{++})$ represents k-splittig of G^{++}.



$spl_k\left(P_4^{++}\right)$: k-splitting of P_4^{++}

Proposition 3.1 *[18] Let G be a* (n,m)*-graph. Then*

$$(i)\ d_{spl_k(G^{++})}(v) = 2(k+1)d_G(v) \quad \text{where} v \in \mathrm{V(G)}$$

$$(ii)\ d_{spl_k(G^{++})}(e) = 2(k+1) \quad \text{where} e \in \mathrm{E(G)}$$

$$(iii)\ d_{spl_k(G^{++})}(v') = 2d_G(v) \quad \text{where v}'\text{is vertex in k} - \text{splitting of } \mathrm{G}^{++} \text{ corrsponding to vertex v in G}$$

$$(iv)\ d_{sp\ k(G^{++})}(e') = 2 \quad \text{where e}' \text{ is vertex in k} - \text{splitting of } \mathrm{G}^{++} \text{ corresponding to edge e in G.}$$

Proposition 3.2 *[18] Let G be a* (n, m)*-graph. Then order and size of* $spl_k(G^{++})$ *are* $(n+m)(k+1)$ *and* $3m(k+1)$.

Theorem 3.3 *Let G be a* (n, m)*-graph. Then*

$$SK_1\big(spl_k(G^{++})\big) = 4(k+1)(3k+1)(SK_1(G) + SK(G)).$$

Proof. Partitioning edge set of $spl_k(G^{++})$ as follows:

$$E(spl_k(G^{++})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$E_1 = \{uv\colon uv \in E(G)\}$

$E_2 = \{ue\colon \text{vertex u in G is incident to edge e in G}\}$

$E_3 = \{uv'\colon \text{vertex u in G is adjacent to vertex v}'\text{in k} - \text{splitting of } \mathrm{G}^{++}\text{corresponding to vertex v in G}\}$

$E_4 = \{ue'\colon \text{vertex u in G is incident to vertex e}'\text{in k} - \text{splitting of } \mathrm{G}^{++}\text{corresponding to edge e in G}\}$

$E_5 = \{u'e\colon \text{vertex u}'\text{in k} - \text{splitting of } \mathrm{G}^{++} \text{ corresponding to vertex u in G incident to edge e in G}\}$

Clearly $|E_1| = m, |E_2| = 2m, |E_3| = 2mk, |E_4| = 2mk$ and $|E_5| = 2mk$

Consider,

$$SK_1\big(spl_k(G^{++})\big) = \frac{1}{2}\sum_{uv\in\left(E\left(sp\ _k(G^{++})\right)\right)} d_{spl_k(G^{++})}(u)\ d_{sp\ k(G^{++})}(v)$$

$$= \frac{1}{2}\sum_{uv\in E_1} d_{spl_k(G^{++})}(u)\ d_{sp\ k(G^{++})}(v) + \frac{1}{2}\sum_{ue\in E_2} d_{spl_k(G^{++})}(u)\ d_{spl_k(G^{++})}(e)$$

$$+\frac{1}{2}\sum_{uv\prime\in E_3} d_{spl_k(G^{++})}(u)\ d_{spl_k(G^{++})}(v') + \frac{1}{2}\sum_{ue\prime\in E_4} d_{spl_k(G^{++})}(u)\ d_{spl_k(G^{++})}$$

$$+\frac{1}{2}\sum_{u\prime e\in E_5} d_{spl_k(G^{++})}(u')\ d_{spl_k(G^{++})}(e)$$

$$=\frac{1}{2}\sum_{uv\in E_1} 2(k+1)d_G(u)2(k+1)d_G(v) + \frac{1}{2}\sum_{ue\in E_2} 2(k+1)d_G(u)2(k+1)$$

$$+\frac{1}{2}\sum_{uv\prime\in E_3} 2(k+1)d_G(u)2d_G(v) + \frac{1}{2}\sum_{ue\prime\in E_4} 2(k+1)d_G(u)2 + \frac{1}{2}\sum_{u\prime e\in E_5} 2d_G(u)2(k+1)$$

$$= 2(k+1)^2 \sum_{uv\in E_1} d_G(u)d_G(v) + 2(k+1)^2 \sum_{ue\in E_2} d_G(u)$$

$$+2(k+1) \sum_{uv\prime\in E_3} d_G(u)d_G(v) + 2(k+1) \sum_{ue\ \in E_4} d_G(u) + 2(k+1) \sum_{u\prime e\in E_5} d_G(u)$$

$$= 2(k+1)^2 \sum_{uv\in E(G)} d_G(u)d_G(v) + 2(k+1)^2 \sum_{u\in V(G)} d_G(u)^2$$

$$+2(k+1)\ \ 2k \sum_{uv\in E(G)} d_G(u)d_G(v) + 2(k+1)\ \ k \sum_{u\in E(G)} d_G(u)^2 + 2(k+1)\ \ k \sum_{u\in E(G)} d_G(u)^2$$

$$= 4(k+1)^2 SK_1(G) + 4\mathrm{SK}(G) + 4k(k+1)2SK_1(G) + 2k(k+1)2\mathrm{SK}(G)$$

$$+4k(k+1)\mathrm{SK}(G)$$

$$= 4(k+1)^2\big(SK_1(G) + \mathrm{SK}(G)\big) + 4k(k+1)\big(SK_1(G) + \mathrm{SK}(G)\big)$$

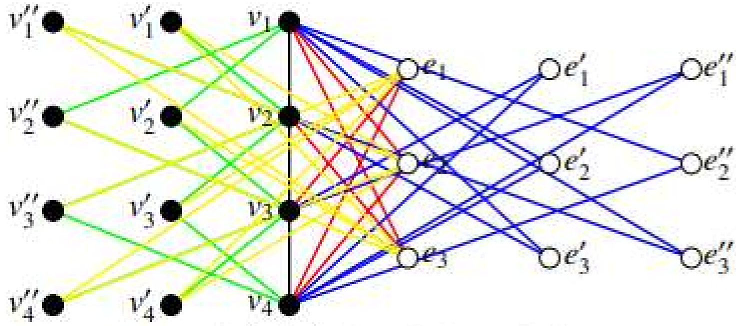
$$= \big(4(k+1)^2 + 4k(k+1)\big)\big(SK_1(G) + \mathrm{SK}(G)\big)$$

$$= 4(k+1)\big((k+1) + 2k\big)\big(SK_1(G) + \mathrm{SK}(G)\big)$$

$$= 4(k+1)(3k+1)\big(SK_1(G) + \mathrm{SK}(G)\big).$$

4 SK_1 Index of $\boldsymbol{spl_k(G^{+-})}$

Let G be a (n, m)-graph and $spl_k(G^{+-})$ represents k-splittig of G^{+-}.



$spl_k\left(P_4^{+-}\right)$: k-splitting of P_4^{+-}

Proposition 4.1 *[18] Let G be a (n, m)-graph. Then*

(i) $d_{sp\ k(G^{+-})}(v) = m(k+1)$ where $v \in V(G)$

(ii) $d_{spl_k(G^{+-})}(e) = (n-2)(k+1)$ where $e \in E(G)$

(iii) $d_{spl_k(G^{+-})}(v') = m$ where v' is vertex in $k-$splitting of G^{+-} corrsponding to vertex v in G

(iv) $d_{spl_k(G^{+-})}(e') = (n-2)$ where $e'$is vertex in $k-$splitting of G^{+-} corresponding to edge e in G.

Proposition 4.2 *[18] Let G be a (n, m)-graph. Then order and size of $spl_k(G^{+-})$ are $(n+m)(k+1)$ and $m(n-1)(2k+1)$.*

Theorem 4.3 *Let G be a (n, m)-graph. Then*

$$M_2\big(spl_k(G^{+-})\big) = m^2(k+1)(3k+1)(m+(n-2)^2).$$

Proof. Partitioning edge set of $spl_k(G^{+-})$ as follows:

$$E(spl_k(G^{+-})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$E_1 = \{uv : uv \in E(G)\}$

$E_2 = \{ue$: vertex u in G is not incident to edge e in G$\}$

$E_3 = \{uv'$: vertex u in G is adjacent to vertex v'in $k-$splitting of G^{+-} due to vertex v in G$\}$

$E_4 = \{ue'$: vertex u in G is incident to vertex e' in $k-$splitting of G^{+-} due to edge e in G$\}$

$E_5 = \{u'e$: vertex u′ in k − splitting of G^{+-} due to vertex u in G notincident to edge e in G$\}$

Clearly, $|E_1| = m, |E_2| = m(n-2), |E_3| = 2mk, |E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$M_2\big(spl_k(G^{+-})\big) = \frac{1}{2}\sum_{uv\in\left(E\left(sp\ _k(G^{+-})\right)\right)} d_{spl_k(G^{+-})}(u)d_{sp\ _k(G^{+-})}(v)$$

$$= \frac{1}{2}\sum_{uv\in E_1} d_{spl_k(G^{+-})}(u)\ d_{spl_k(G^{+-})}(v) + \frac{1}{2}\sum_{ue\in E_2} d_{sp\ _k(G^{+-})}(u)\ d_{spl_k(G^{+-})}(e)$$

$$+\frac{1}{2}\sum_{uv\prime\in E_3} d_{spl_k(G^{+-})}(u)\ d_{spl_k(G^{+-})}(v') + \frac{1}{2}\sum_{ue\ \in E_4} d_{sp\ _k(G^{+-})}(u)\ d_{spl_k(G^{+-})}(e')$$

$$+\frac{1}{2}\sum_{u\prime e\in E_5} d_{spl_k(G^{+-})}(u')\ d_{sp\ _k(G^{+-})}(e)$$

$$= \frac{1}{2}\sum_{uv\in E_1} m(k+1)\ m(k+1) + \frac{1}{2}\sum_{ue\in E_2} m(k+1)\ (n-2)(k+1) + \frac{1}{2}\sum_{uv\prime\in E_3} m(k+1)\ m$$

$$+\frac{1}{2}\sum_{ue\ \in E_4} m(k+1)\ (n-2) + \frac{1}{2}\sum_{u\prime e\in E_5} m\ (n-2)(k+1)$$

$$= \frac{1}{2}m^2(k+1)^2\sum_{uv\in E_1} 1 + \frac{1}{2}m(n-2)(k+1)^2\sum_{ue\in E_2} 1 + \frac{1}{2}m^2(k+1)\sum_{uv\prime\in E_3} 1$$

$$+\frac{1}{2}m(n-2)(k+1)\sum_{ue\prime\in E_4} 1 + \frac{1}{2}m(n-2)(k+1)\sum_{u\prime e\in E_5} 1$$

$$= \frac{1}{2}m^2(k+1)^2m + \frac{1}{2}m(n-2)(k+1)^2m(n-2) + \frac{1}{2}m^2(k+1)2mk$$

$$+\tfrac{1}{2}m(n-2)(k+1)mk(n-2) + \tfrac{1}{2}m(n-2)(k+1)mk(n-2)$$

$$= \frac{1}{2}m^3(k+1)^2 + \frac{1}{2}m^2(n-2)^2(k+1)^2 + \frac{1}{2}2m^3k(k+1) + \frac{1}{2}2m^2(n-2)^2k(k+1)$$

$$= \tfrac{1}{2}(m^3 + m^2(n-2)^2)(k+1)^2 + (m^3 + m^2(n-2)^2)k(k+1)$$

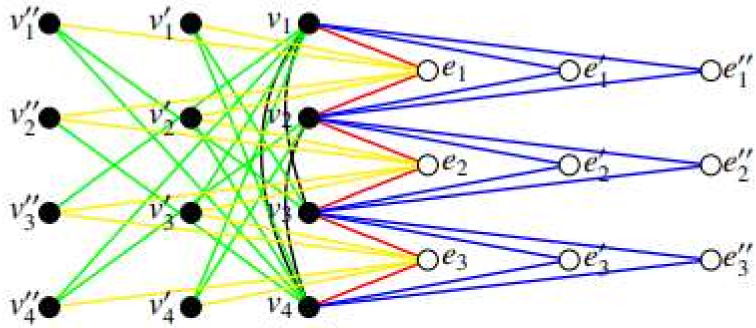
$$= \tfrac{1}{2}(m^3 + m^2(n-2)^2)((k+1)^2 + k(k+1))$$

$$= \frac{1}{2}(k+1)(3k+1)(m^3 + m^2(n-2)^2)$$

$$= \frac{1}{2}m^2(k+1)(3k+1)(m + (n-2)^2).$$

5 SK1 Index of $spl_k(G^{-+})$

Let G be a (n,m)-graph and $spl_k(G^{-+})$ represents k-splittig of G^{-+}.



$spl_k\left(P_4^{-+}\right)$: k-splitting of P_4^{-+}

Proposition 5.1 *[18] Let G be a (n,m)-graph. Then*

$$(i)\ d_{sp\ k(G^{-+})}(v) = (n-1)(k+1) \quad \text{where} v \in \mathrm{V(G)}$$

$$(ii)\ d_{sp\ k(G^{-+})}(e) = 2(k+1) \quad \text{where} e \in \mathrm{E(G)}$$

$$(iii)\ d_{sp\ k(G^{-+})}(v') = (n-1) \quad \text{where v}'\text{is vertex in} k - \text{splitting of } \mathrm{G}^{-+} \text{ due to vertex v in G}$$

$$(iv)\ d_{spl_k(G^{-+})}(e') = 2 \quad \text{where e}' \text{ is vertex in k} - \text{splitting of } \mathrm{G}^{-+} \text{ due to edge e in G.}$$

Proposition 5.2 *[18] Let G be a (n,m)-graph. Then order and size of $spl_k(G^{-+})$ are $(n+m)(k+1)$ and $\frac{1}{2}[n(n-1)+2m](2k+1)$.*

Theorem 5.3 *Let G be a (n,m)-graph. Then*

$$SK_1\big(spl_k(G^{-+})\big) = \frac{1}{2}(n-1)(k+1)(3k+1)\left((n-1)\left(\frac{n(n-1)}{2} - m\right) + 4m\right).$$

Proof. Partitioning edge set of $spl_k(G^{-+})$ as follows:

$$E(spl_k(G^{-+})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$E_1 = \{uv : uv \notin E(G)\}$$

$E_2 = \{ue\text{: vertex u in G is incident to edge e in G}\}$

$E_3 = \{uv'\text{: vertex u in G is not adjacent to vertex v}' \text{ ink} - \text{splitting of } G^{-+} \text{ corresponding to vertex v in G}\}$

$E_4 = \{ue'\text{: vertex u in G is incident to vertex e}' \text{ in k} - \text{splitting of } G^{-+} \text{ corresponding to edge e in G}\}$

$E_5 = \{u'e\text{: vertex u}' \text{ in k} - \text{splitting of } G^{-+} \text{ corresponding to vertex u in G is not incident to edge e in G}\}$

Clearly, $|E_1| = \frac{n(n-1)}{2} - m, |E_2| = 2m, |E_3| = 2k\left(\frac{n(n-1)}{2} - m\right), |E_4| = 2mk$ and $|E_5| = 2mk$.

Consider,

$$SK_1(spl_k(G^{-+})) = \frac{1}{2} \sum_{uv\in\left(E(sp\ _k(G^{-+}))\right)} d_{sp\ _k(G^{-+})}(u) d_{spl_k(G^{-+})}(v)$$

$$= \frac{1}{2} \sum_{uv\in E_1} d_{spl_k(G^{-+})}(u)\ d_{sp\ _k(G^{-+})}(v) + \frac{1}{2} \sum_{ue\in E_2} d_{sp\ _k(G^{-+})}(u)\ d_{sp\ _k(G^{-+})}(e)$$

$$+ \frac{1}{2} \sum_{uv\ \in E_3} d_{sp\ _k(G^{-+})}(u)\ d_{sp\ _k(G^{-+})}(v') + \frac{1}{2} \sum_{ue\ \in E_4} d_{sp\ _k(G^{-+})}(u)\ d_{sp\ _k(G^{-+})}(e')$$

$$+ \frac{1}{2} \sum_{u'e\in E_5} d_{sp\ _k(G^{-+})}(u')\ d_{sp\ _k(G^{-+})}(e)$$

$$= \frac{1}{2} \sum_{uv\in E_1} (n-1)(k+1)(n-1)(k+1) + \frac{1}{2} \sum_{ue\in E_2} (n-1)(k+1)2(k+1)$$

$$+ \frac{1}{2} \sum_{uv\ \in E_3} (n-1)(k+1)(n-1) + \frac{1}{2} \sum_{ue'\in E_4} (n-1)(k+1)2 + \frac{1}{2} \sum_{u'e\in E_5} (n-1)2(k+1)$$

$$= \frac{1}{2}(n-1)^2(k+1)^2 \sum_{uv\in E_1} 1 + \frac{1}{2}(2)(n-1)(k+1)^2 \sum_{ue\in E_2} 1 + \frac{1}{2}(n-1)^2(k+1) \sum_{uv'\in E_3} 1$$

$$+ \frac{1}{2}(2)(n-1)(k+1) \sum_{ue'\in E_4} 1 + \frac{1}{2}(2)(n-1)(k+1) \sum_{u'e\in E_5} 1$$

$$= \frac{1}{2}(n-1)^2(k+1)^2 m' + \frac{1}{2}(n-1)(k+1)^2 4m + \frac{1}{2}(n-1)^2(k+1)2km'$$

$$+\frac{1}{2}(n-1)(k+1)4mk+\frac{1}{2}(n-1)(k+1)4mk$$

$$=\frac{1}{2}(n-1)^2(k+1)\big((k+1)+2k\big)m'+4m(n-1)(k+1)\big((k+1)+2k\big)$$

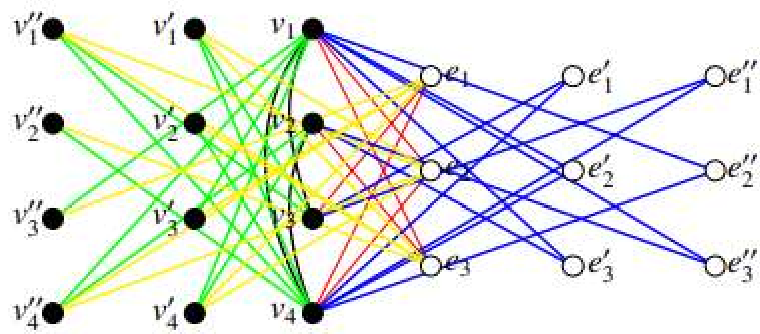
$$=\big((n-1)^2(k+1)m'+4m(n-1)(k+1)\big)(3k+1)$$

$$=\frac{1}{2}(n-1)(k+1)(3k+1)\big((n-1)m'+4m\big)$$

$$=\frac{1}{2}(n-1)(k+1)(3k+1)\left((n-1)\left(\frac{n(n-1)}{2}-m\right)+4m\right).$$

6 SK1 Index of $spl_k(G^{--})$

Let G be a (n,m)-graph and $spl_k(G^{--})$ represents k-splittig of G^{--}.



$spl_k\left(P_4^{--}\right)$: k-splitting of P_4^{--}

Proposition 6.1 *[18] Let G be a (n,m)-graph. Then*

(i) $d_{spl_k(G^{--})}(v)=\big(n+m-1-2d_G(v)\big)(k+1)$ where $v \in V(G)$

(ii) $d_{spl_k(G^{--})}(e)=(n-2)(k+1)$ where $e \in E(G)$

(iii) $d_{spl_k(G^{--})}(v')=\big(n+m-1-2d_G(v)\big)$ where v' is vertex in $k-$ splitting of G^{--} corresponding to vertex v in G

(iv) $d_{spl_k(G^{--})}(e')=(n-2)$ where e' is vertex in $k-$ splitting of G^{--} corresponding to edge e in G.

Proposition 6.2 *[18] Let G be a (n,m)-graph. Then order and size of $spl_k(G^{--})$ are $(n+m)(k+1)$ and $\frac{1}{2}[n(n-1)+2m(n-3)](2k+1)$.*

Theorem 6.3 *Let G be a (n,m)-graph. Then*

$SK_1(spl_k(G^{--})) = \frac{1}{2}(k+1)(3k+1)[(n+m-1)^2\left(\frac{n(n-1)}{2}-m\right)-(n+m-1)\overline{M_1}(G)+2\overline{M_2}(G)+\frac{1}{2}(n+m-1)m(n-2)-(n-1)2m+4\mathrm{SK}(G)]$.

Proof. Partitioning edge set of $spl_k(G^{--})$ as follows:

$$E(spl_k(G^{--})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$E_1 = \{uv: uv \notin E(G)\}$

$E_2 = \{ue:$ vertex u in G is not incident to edge e in G$\}$

$E_3 = \{uv':$ vertex u in G is not adjacent to vertex v′in k − splitting of G^{--}corresponding to vertex v in G$\}$

$E_4 = \{ue':$ vertex u in G is not incident to vertex e′in k − splitting of G^{--} corresponding to edge e in G$\}$

$E_5 = \{u'e:$ vertexu′ink − splitting of G^{--} corresponding to vertex u in G is not incident to edge e in G$\}$

Clearly, $|E_1| = \frac{n(n-1)}{2}-m, |E_2| = m(n-2), |E_3| = 2k\left(\frac{n(n-1)}{2}-m\right), |E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$SK_1(spl_k(G^{--})) = \frac{1}{2}\sum_{uv\in\left(E(spl_k(G^{--}))\right)} d_{sp\ k(G^{--})}(u)d_{spl_k(G^{--})}(v)$$

$$= \frac{1}{2}\sum_{uv\in E_1} d_{spl_k(G^{--})}(u)\ d_{spl_k(G^{--})}(v) + \frac{1}{2}\sum_{ue\in E_2} d_{sp\ k(G^{--})}(u)\ d_{spl_k(G^{--})}(e)$$

$$+\frac{1}{2}\sum_{uv'\in E_3} d_{spl_k(G^{--})}(u)\ d_{sp\ k(G^{--})}(v') + \frac{1}{2}\sum_{ue'\in E_4} d_{spl_k(G^{--})}(u)\ d_{sp\ k(G^{--})}(e')$$

$$+\frac{1}{2}\sum_{u'e\in E_5} d_{spl_k(G^{--})}(u')\ d_{spl_k(G^{--})}(e)$$

$$= \frac{1}{2}\sum_{uv\in E_1}(n+m-1-2d_G(u))(k+1)(n+m-1-2d_G(v))(k+1)$$

$$+\frac{1}{2}\sum_{ue\in E_2}(n+m-1-2d_G(u))(n-2)(k+1)$$

$$+\frac{1}{2}\sum_{uv\prime\in E_3}(n+m-1-2d_G(u))(k+1)(n+m-1-2d_G(v))$$

$$+\frac{1}{2}\sum_{ue\prime\in E_4}(n+m-1-2d_G(u))(k+1)(n-2)+\frac{1}{2}\sum_{u\prime e\in E_5}(n+m-1-2d_G(u))(n-2)(k+1)$$

$$=\frac{1}{2}(k+1)^2\sum_{uv\in E_1}[(n+m-1)^2-2(n+m-1)(d_G(u)+d_G(v))+4d_G(u)d_G(v)]$$

$$+\frac{1}{2}(n-2)(k+1)^2\sum_{ue\in E_2}(n+m-1-2d_G(u))$$

$$+\frac{1}{2}(k+1)\sum_{uv\ \in E_3}[(n+m-1)^2-2(n+m-1)(d_G(u)+d_G(v))+4d_G(u)d_G(v)]$$

$$+\frac{1}{2}(n-2)(k+1)\sum_{ue\prime\in E_4}(n+m-1-2d_G(u))+\frac{1}{2}(n-2)(k+1)\sum_{u\prime e\in E_5}(n+m-1-2d_G(u))$$

$$=\frac{1}{2}(k+1)^2\left[(n+m-1)^2\sum_{uv\in E_1}1-2(n+m-1)\sum_{uv\in E_1}(d_G(u)+d_G(v))+4\sum_{uv\in E_1}d_G(u)d_G(v)\right]$$

$$+\frac{1}{2}(n-2)(k+1)^2\left[(n+m-1)\sum_{ue\in E_2}1-2\sum_{ue\in E_2}d_G(u)\right]$$

$$+\frac{1}{2}(k+1)\left[(n+m-1)^2\sum_{uv\ \in E_3}1-2(n+m-1)\sum_{uv\prime\in E_3}(d_G(u)+d_G(v))+4\sum_{uv\ \in E_3}d_G(u)d_G(v)\right]$$

$$+\frac{1}{2}(n-2)(k+1)\left[(n+m-1)\sum_{ue\prime\in E_4}1-2\sum_{ue\prime\in E_4}d_G(u)\right]$$

$$+\frac{1}{2}(n-2)(k+1)\left[(n+m-1)\sum_{u\prime e\in E_5}1-2\sum_{u\prime e\in E_5}d_G(u)\right]$$

$$SK_1\big(spl_k(G^{--})\big)=$$

$$\frac{1}{2}(k+1)^2\left[(n+m-1)^2\sum_{uv\notin E(G)}1-2(n+m-1)\sum_{uv\notin E(G)}(d_G(u)+d_G(v))\right.$$
$$\left.+4\sum_{uv\notin E(G)}d_G(u)d_G(v)\right]$$
$$+\frac{1}{2}(n-2)(k+1)^2\left[(n+m-1)\sum_{ue\in E_2}1-2\sum_{u\in V(G)}(n-1-d_G(u))d_G(u)\right]$$
$$+\frac{1}{2}(k+1)\left[(n+m-1)^2\sum_{uv\ \in E_3}1-2(n+m-1)(2k)\sum_{uv\notin E(G)}(d_G(u)+d_G(v))\right.$$
$$\left.+4(2k)\sum_{uv\notin E(G)}d_G(u)d_G(v)\right]$$
$$+\frac{1}{2}(n-2)(k+1)\left[(n+m-1)\sum_{ue'\in E_4}1-2k\sum_{u\in V(G)}(n-1-d_G(u))d_G(u)\right]$$
$$+\frac{1}{2}(n-2)(k+1)\left[(n+m-1)\sum_{u'e\in E_5}1-2k\sum_{u\in V(G)}(n-1-d_G(u))d_G(u)\right]$$
$$=\frac{1}{2}(k+1)^2\left[(n+m-1)^2\left(\frac{n(n-1)}{2}-m\right)-2(n+m-1)\overline{M_1}(G)+4\overline{M_2}(G)\right]$$
$$+(n-2)(k+1)^2[(n+m-1)m(n-2)-2(n-1)2m+2M_1(G)]$$
$$+(k+1)\left[(n+m-1)2k\left(\tfrac{n(n-1)}{2}-m\right)-4k(n+m-1)\overline{M_1}(G)+8k\overline{M_2}(G)\right]$$
$$+(n-2)(k+1)[(n+m-1)mk(n-2)-2k(n-1)2m+2kM_1(G)]$$
$$+(n-2)(k+1)[(n+m-1)mk(n-2)-2k(n-1)2m+2kM_1(G)]$$
$$=\frac{1}{2}(k+1)(3k+1)\left[(n+m-1)^2\left(\frac{n(n-1)}{2}-m\right)-2(n+m-1)\overline{M_1}(G)+4\overline{M_2}(G)\right]$$
$$+(k+1)(3k+1)[(n+m-1)m(n-2)-2(n-1)2m+2M_1(G)]$$
$$=\frac{1}{2}(k+1)(3k+1)[(n+m-1)^2\left(\frac{n(n-1)}{2}-m\right)-(n+m-1)\overline{M_1}(G)+2\overline{M_2}(G)$$

$$+\frac{1}{2}(n+m-1)m(n-2)-(n-1)2m+4\mathrm{SK}(G)]$$

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