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An Algebraic Study of Non commutative Riemann Surfaces: Deformation, Sheaf-Theoretic Structures, and Morita Equivalence

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ABSTRACT

This paper proposes an algebraic framework for the study of noncommutative Riemann surfaces as deformation-type and categorical-type analogues of classical compact Riemann surfaces. The geometric language of points, coordinate charts, holomorphic functions, and sheaves is structurally incomplete when one does not use a commutative algebra of coordinate functions as the underlying algebra. The study thus considers a noncommutative Riemann surface as a controlled algebraic object given by a formal deformation algebra, a category of modules and a sheaf-theoretic substitute by quotients or localization. The study constructs a formal noncommutative coordinate algebra Aθ that arises from a commutative algebra of functions, investigates the first-order commutator as the obstruction to commutativity and interprets geometric structure by means of modules and bimodules rather than by pointwise evaluation of functions. Since equivalent categories of modules can preserve geometric information even when the defining algebras are not isomorphic, Morita equivalence is the main classification tool. The article shows that the best way to picture noncommutative Riemann surfaces is not as actual topological surfaces with coordinates that do not commute but as algebraic-categorical spaces whose geometric content

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lies in deformation, coherence, equivalence and representation-theoretic data.

1. Introduction

A compact Riemann surface is classically of central importance in complex analysis, topology and algebraic geometry. It is a connected complex manifold of dimension one; and, when compact, can also be considered a smooth projective algebraic curve over the complex numbers. The bridge between analytic and algebraic descriptions is one reason Riemann surfaces are basic objects in modern geometry. The role of analytic objects, like meromorphic functions, divisors and (holomorphic) line bundles, and differentials, is taken over in algebraic geometry by their algebraic counterparts: function fields, coordinate rings, coherent sheaves, and algebraic curves. A surface is not just a topological space with complex charts; it is a certain geometric object whose structure can be reconstructed from its functions and sheaves.

The noncommutative geometry grows from distinct but related observation; in many geometric and physical situations, the algebra of functions is a more flexible object than the underlying set of points. Commutative algebras of functions arise from ordinary spaces while quantum or deformed spaces produce noncommutative algebras. The noncommutative algebra is considered a coordinate algebra of a generalized space. The philosophy is especially useful in noncommutative projective geometry in which one replaces a projective scheme with a quotient category of graded modules and in the theory of noncommutative schemes where equivalences of sheaf categories are viewed to be the same as geometric equivalences (Burban & Drozd, 2022; Mori & Ueyama, 2021).

The document explains the title noncommutative Riemann surface, to have an operational and precise sense. It does not presume that this name is assigned to an object universally agreed upon. The algebraic data that should replace a compact Riemann surface when one relaxes to the non-commutativity of the coordinate algebra. A noncommutative Riemann surface is modelled here by three linked structures. The first structure is a deformation algebra $A\theta$ wherein its product is a formal or parameter-dependent perturbation of the ordinary product of functions. The second structure is a category of modules over $A\theta$ that serves the same role as sheaves on the surface. Finally, we have a Morita-equivalence class that records when two noncommutative coordinate systems define the same categorical geometry. The approach is aligned with Hu et al. (2023) and Itaba and Mori (2023), which focuses on noncommutative projective spaces, noncommutative curves, quantum projective planes, noncommutative conics, and projective noncommutative surfaces.

A mathematical warning is needed at the start. For a smooth curve that is complex and one-dimensional, the bivectors are holomorphic but they do vanish. A nontrivial holomorphic Poisson deformation of the sheaf of holomorphic functions is thus unavailable like it is for complex manifolds of dimension higher than two. The noncommutative study is not empty, though. That is, the suitable framework should differentiate smooth deformation quantizations of the real two-dimensional surface, algebraic deformations of graded coordinate rings, noncommutative projective curve categories and categorical models arising from orders or module categories. Mathematical work resulting in precisely this categorical and algebraic change is encountered, especially along the lines developing Morita theory for noncommutive noetherian schemes as well as the noncommutative projective surfaces and quantum projective planes (Burban & Drozd, 2022; Gammelgaard & Gyenge, 2024; Mori & Nyman, 2021).

The structural deficiency rather than the computational one is the research gap this paper is covering. The noncommutative algebraic geometry has many power constructions. However, the algebraic interpretation of noncommutative analogue of Riemann surfaces is spread out over the deformation theory, module theory, projective noncommutative schemes and Morita equivalence.

The present paper synthesizes these components in an algebraic framework. The contribution of this paper is to present a generalisation of a Riemann surface by replacing point-based geometry with algebraic deformation, sheaf categories with module categories and isomorphism of spaces with equivalence of categories. The paper is theoretical and contains definitions and construction, propositions and categorical comparisons, not empirical or numerical experiments.

2. Research Objectives

To examine the algebraic foundations through which a classical compact Riemann surface may be interpreted in a noncommutative setting.

To analyse how commutative coordinate algebras associated with Riemann surfaces can be replaced by formal deformation algebras or graded noncommutative coordinate rings.

To study module-theoretic and sheaf-theoretic interpretations of noncommutative Riemann surfaces, especially the replacement of ordinary coherent sheaves by categories of modules.

To investigate the role of Morita equivalence in determining when distinct noncommutative algebras define equivalent geometric structures.

To identify how noncommutative Riemann surfaces extend classical algebraic curve theory while preserving categorical features such as coherence, duality, and equivalence.

3. Research Questions

How can the algebraic structure of a compact Riemann surface be represented when its coordinate algebra is deformed into a noncommutative algebra?

What replaces coordinate rings, sheaves, and morphisms when the ordinary point-set geometry of a Riemann surface is no longer primary?

How does the first-order commutator of a deformation product measure the departure from classical commutativity?

What role does Morita equivalence play in comparing noncommutative Riemann surfaces whose coordinate algebras are not isomorphic?

Can noncommutative Riemann surfaces be treated as categorical analogues of algebraic curves rather than as literal complex manifolds with noncommuting coordinates?

4. Literature Review

Recent research in noncommutative algebraic geometry has emphasized categorical foundations rather than those of the point-set. The main reference for this paper is (Burban, Drozd 2022), where a Morita theory for noncommutative noetherian schemes is worked out. This theory studies equivalences of categories of quasi-coherent sheaves and derives criteria for reduced noncommutative curves. Their results back the claim that the equivalence of sheaf-like categories is not just an auxiliary algebraic fact. It is a genuine geometric classification principle. To study noncommutative Riemann surfaces one has to think either that the object is not classified only by an algebra A, but by the module and sheaf categories associated to A.

One more related stream of work deals with quantum projective spaces with categorical characterizations. Mori and Ueyama (2021) describe quantum projective spaces via abelian categories, which shows that noncommutative projective geometry can be formulated without an ordinary space. The significance of their work for the present paper is that it enhances the categorical reading of geometry. Mori and Nyman (2021) develop local duality for connected Z-algebras. The results provide homological tools which are useful when noncommutative projective objects are described by graded or indexed algebras. To recapitulate, these studies show that algebraic geometry can be reconstructed via categories, duality, regularity and finiteness.

Contemporary instances of noncommutative algebraic geometry can be found in noncommutative projective planes and conics. Itaba and Mori (2023) study the quantum projective planes which are finite over their centers and relate geometric properties to finite order automorphism data. Moreover, central finiteness is identified. In Hu, Matsuno, and Mori (2023), the noncommutative conics in Calabi-Yau quantum projective planes are classified, giving a concrete example of how familiar one-dimensional projective objects can be studied in noncommutative ambient spaces. Hu and Mori (2024) show further study of noncommutative hypersurfaces within the context of Clifford quadratic complete intersection The aforementioned works aren't classical analytic Riemman-surface papers, but show how to set algebraic models for noncommutative curve-like objects.

The paper is also informed by recent work on noncommutative projective surfaces and moduli spaces. Gammelgaard and Gyenge (2024) are introducing projective noncommutative surfaces associated to quiver varieties and framed torsion-free sheaves. Projective noncommutative geometry interfaces moduli and representation theory in noncommutative projective geometry. The paper of Abdelgadir, Okawa and Ueda (2024) constructs compact moduli of marked noncommutative cubic surfaces. The paper of Baumann, Belmans and van Garderen (2024) studies central curves on noncommutative surfaces and relates orders with stacky geometric structures. As a result, Noncommutative geometry in low dimensions may be moving toward a synthesis of orders, stacks, sheaves, moduli, and categorical equivalence.

Relevant are deformation and differential-geometric views. Wang and Wu (2022) construct Riemannian geometry for noncommutative super surfaces which shows the metrics, connections and curvatures etc. can be conveniently done for noncommutative surface like objects Nikolaev (2021) draws connections between elliptic surfaces, noncommutative tori and continued fractions suggesting a correspondence between arithmetic geometry and operator-algebraic noncommutative geometry. In Rains (2025), we develop noncommutative projective surfaces in connection with difference equations. This shows that noncommutative surfaces have unexpected properties, including the ability to encode integrable systems and moduli of sheaves. All of these works justify our choice in this paper to consider noncommutative Riemann surfaces as algebraic-categorical structures rather than as one fixed class of objects.

Previous studies do not address. The noncommutative world has strong tools for noncommutative schemes, projective spaces, conics, quantum planes, moduli, and deformations. But the algebraic study of noncommutative Riemann surfaces requires an explicit synthesis of deformation algebras, module categories, sheaf replacements, and Morita equivalence. A controlled model is put forward with this paper

which also clears what we can and cannot expect from a noncommutative analogue of a one-dimensional complex surface.

5. Mathematical Preliminaries

A Riemann surface is a connected complex manifold of complex dimension one. Locally it is modelled on open subsets of the complex plane, and its transition functions are holomorphic. A compact Riemann surface carries enough meromorphic functions to be viewed algebraically as a smooth projective curve over the complex numbers. In this paper, the phrase classical Riemann surface refers primarily to this compact algebraic situation, because the algebraic curve viewpoint is the most suitable starting point for noncommutative generalization.

An algebraic curve over C is a one-dimensional algebraic variety or scheme. In affine form, a curve is studied through a coordinate ring C[X], and in projective form it is studied through a homogeneous coordinate ring and coherent sheaves. The commutative coordinate ring is not merely a collection of functions; it encodes the geometry of the curve. Prime ideals correspond to points in the affine case, and coherent sheaves encode vector bundles, divisors, line bundles, and torsion phenomena. Noncommutative geometry keeps the algebraic role of functions but weakens commutativity.

A noncommutative algebra over C is an associative C-algebra A in which ab need not equal ba. The commutator [a,b] = ab - ba measures the failure of commutativity. If A is interpreted as a coordinate algebra, then the ordinary pointwise interpretation of functions is lost or at least weakened, because evaluation at points is no longer the controlling mechanism. Instead, representations of A, modules over A, and categories built from modules become the main geometric carriers.

A formal deformation of a commutative algebra A0 is a C[[θ]]-module Aθ = A0[[θ]] equipped with an associative product $*_\theta$ such that $f *_\theta g = fg$ modulo θ. One usually writes $f *_\theta g = fg + \theta C1(f,g) + \theta^2 C2(f,g) + \cdots$, where the maps Ci are bilinear operations on A0. The first-order commutator is $[f,g]_\theta = f *_\theta g - g *_\theta f$. If C1(f,g) - C1(g,f) is nonzero, the deformation is noncommutative to first order. In smooth symplectic settings, such first-order data are related to Poisson brackets, but in the holomorphic one-dimensional case one must be careful because holomorphic bivectors vanish.

A sheaf on a topological space assigns algebraic data to open sets in a way compatible with restriction and gluing. Coherent sheaves on an algebraic curve form an abelian category that is central to curve theory. In noncommutative projective geometry, sheaves are frequently replaced by categories of modules or quotient categories of graded modules. If A is a suitable graded algebra, one considers categories such as Gr(A),

gr(A), tors(A), qgr(A), or QGr(A), where torsion modules are quotiented out to obtain a category behaving like coherent or quasi-coherent sheaves on a projective noncommutative scheme. This categorical replacement is central in recent work on quantum projective spaces and noncommutative projective schemes (Mori & Ueyama, 2021; Mizuno, 2024a).

A left A-module is an abelian group M with an action $A \times M \to M$, while a right A-module has an action $M \times A \to M$. When A is noncommutative, left and right modules are generally different. A bimodule is simultaneously a left module over one algebra and a right module over another, with compatible actions. Bimodules are important because they define functors between module categories and therefore act as correspondences between noncommutative spaces.

Morita equivalence is the relation between two algebras A and B whose module categories are equivalent. In ordinary commutative geometry, isomorphism of coordinate rings is often the natural algebraic counterpart of isomorphism of spaces. In noncommutative geometry, Morita equivalence is more flexible and often more geometric. For instance, A and the matrix algebra $M_n(A)$ are usually Morita equivalent even though they are not isomorphic as algebras when $n > 1$. Burban and Drozd (2022) show the importance of this idea for noncommutative noetherian schemes and noncommutative curves.

A noncommutative projective scheme is not defined by constructing an ordinary topological Proj in the same way as in commutative algebraic geometry. Instead, it is often defined through an appropriate quotient category of graded modules. If A is a connected graded noetherian algebra, the category qgr(A) = gr(A)/tors(A) serves as the category of coherent sheaves on the projective noncommutative object. This construction is a guiding model for the sheaf-theoretic part of the present paper.

6. Methodology

Theory, algebra, and category are the methodologies. The study did not make use of empirical data, simulation, statistical testing, or computational experiments. The approach we will follow is to begin with the commutative algebraic description of a compact Riemann surface and then invent noncommutative replacements for the main structures: multiplication, sheaves, morphisms and equivalence.

The subsequent analysis in not numerical but structural.

Algebraic extraction is the first step. From a compact Riemann surface X, one considers algebras naturally associated with X, for example, the algebra of smooth complex valued functions $C^\infty(X)$, the field of meromorphic functions C(X), or a graded homogeneous coordinate algebra associated with an algebraic

embedding of X. These algebras serve different purposes. The smooth algebra is suitable for deformation quantization, the meromorphic function field records birational data, and the graded coordinate algebra connects the surface to projective noncommutative geometry.

The next phase involves deformation. A formal parameter θ is introduced and ordinary multiplication is replaced by a product *θ. The formal deformation algebra Aθ = A0[[θ]], where A0 is a commutative algebra associated with X, is produced. The paper offers no proof that every compact Riemann surface admits a nontrivial holomorphic deformation of its structure sheaf. Instead, it assumes that noncommutativity is represented algebraically as deformation at the level of smooth algebras or graded coordinate algebras.

The next step is categorical reconstruction. Noncommutativity weakens the usual interpretation of what a point is, so geometric structure is read via the relevant module category Mod(Aθ), the category of finitely generated modules, and projective quotient categories when a grading is available. The present procedure basically follows the spirit of noncommutative projective geometry where categories of sheaves are replaced by categories of modules (Mori and Nyman 2021; Mori and Ueyama 2021).

Classifying by equivalence is the fourth step. Rather than saying that two noncommutative Riemann surfaces are equal precisely when the algebras that define them are isomorphic, we compare the categories of modules that they define. If Aθ and Bη have equivalent categories of modules or coherent sheaf analogues then they are Morita-equivalent noncommutative geometric objects. This is in agreement with the modern Morita theory for noncommutative schemes (Burban & Drozd, 2022).

7. Algebraic Construction of Noncommutative Riemann Surfaces

Let X be a compact Riemann surface and let A0 denote a commutative C-algebra naturally associated with X. Depending on the chosen setting, A0 may be C∞(X), the algebra of smooth complex-valued functions; a subalgebra of meromorphic functions; or a graded coordinate algebra of the corresponding smooth projective curve. The present construction is formal and algebraic: it begins with A0 and introduces a noncommutative product while preserving A0 as the classical limit.

Definition 1 (formal noncommutative coordinate algebra). A formal noncommutative coordinate algebra associated with X is a pair (Aθ, ∗θ), where Aθ = A0[[θ]] as a C[[θ]]-module and ∗θ is a C[[θ]]-bilinear associative product satisfying f ∗θ g ≡ fg modulo θ for all f,g ∈ A0. The parameter θ records the deformation. The classical surface is recovered at θ = 0.

The product is written as

$f *\theta g = fg + \theta C1(f,g) + \theta^2 C2(f,g) + \theta^3 C3(f,g) + \cdots,$

where each Ci: A0 × A0 → A0 is bilinear. The deformation is noncommutative if there exist f,g ∈ A0 such that $f *\theta g \neq g *\theta f$. Its first-order commutator is

$[f,g]\theta = f *\theta g - g *\theta f = \theta(C1(f,g) - C1(g,f)) + O(\theta^2).$

Proposition 1 (first-order obstruction to commutativity). Let (Aθ, ∗θ) be a formal deformation of a commutative algebra A0. If C1 is symmetric, then the commutator [f,g]θ has no first-order term. If C1(f,g) - C1(g,f) is nonzero for some f,g, then Aθ is noncommutative to first order.

Proof. By definition, $f *\theta g = fg + \theta C1(f,g) + O(\theta^2)$ and $g *\theta f = gf + \theta C1(g,f) + O(\theta^2)$. Since A0 is commutative, fg = gf. Subtracting the two expansions gives $[f,g]\theta = \theta(C1(f,g) - C1(g,f)) + O(\theta^2)$. The two assertions follow immediately from this formula.

This proposition is elementary, but it identifies the algebraic meaning of deformation. The noncommutative Riemann surface is not obtained by placing noncommuting points on X. It is obtained by modifying multiplication in the algebra associated with X so that the classical algebra remains visible as the quotient Aθ/(θ). The commutator is the algebraic shadow of the lost commutativity.

Definition 2 (noncommutative Riemann surface, deformation model). A deformation-model noncommutative Riemann surface is a triple Xθ = (Aθ, Cθ, π), where Aθ is a formal noncommutative coordinate algebra with classical limit A0, Cθ is a chosen sheaf-like or module-theoretic category associated with Aθ, and π: Aθ → A0 is the specialization map modulo θ. Two such triples are equivalent if their associated categories Cθ are equivalent in a way compatible with the specialization to the classical category.

This definition deliberately includes the category Cθ. A noncommutative algebra alone rarely contains all the geometric information in a transparent form. In projective settings, for example, the category qgr(A) is often more geometric than A itself. Recent categorical work on quantum projective spaces and noncommutative projective schemes strengthens this viewpoint (Mori & Ueyama, 2021; Mizuno, 2024a).

A useful example is provided by a quantum torus. Let q ∈ C× and let Aq be the algebra generated by U and V subject to the relation VU = qUV. If q = 1, the algebra is commutative and resembles a Laurent polynomial coordinate algebra on an algebraic torus. If q ≠ 1, the generators fail to commute. When $q = \exp(2\pi i\theta)$, the parameter θ measures the deformation. This example is most naturally associated with elliptic and toroidal geometries rather than arbitrary compact Riemann surfaces, but it illustrates the central

principle: a geometric object may be studied through a noncommutative coordinate algebra whose classical limit is commutative. Nikolaev (2021) uses noncommutative tori in the study of elliptic surfaces, while work on quantum projective planes and conics shows how similar algebraic principles appear in projective noncommutative geometry (Hu et al., 2023; Itaba & Mori, 2023).

For higher-genus Riemann surfaces, a direct quantum-torus model is insufficient. The algebraic construction must use either smooth deformation quantization, orders over curves or surfaces, or projective categorical replacements. This is why the present paper does not define noncommutative Riemann surfaces only by generators and relations. A generator-relation presentation is useful when available, but the broader object is categorical: it consists of an algebra together with the sheaf-like module category that carries coherent geometry.

Definition 3 (projective categorical model). Let $A = \bigoplus_{n\geq 0} A_n$ be a connected graded right noetherian C-algebra. Let $gr(A)$ be the category of finitely generated graded right A-modules and $tors(A)$ the full subcategory of finite-dimensional torsion modules. The projective categorical model associated with A is $qgr(A) = gr(A)/tors(A)$. A noncommutative Riemann surface in the projective categorical sense is a $qgr(A)$ whose homological and dimensional behaviour is curve-like and whose commutative specialization recovers coherent sheaves on a smooth projective curve.

This definition connects the present study to noncommutative projective geometry. Conner and Goetz (2023) examine quantum projective planes through graded twisted tensor products, while Hu et al. (2023) and Mizuno (2024b) study projective noncommutative objects through coordinate algebras and quotient categories. The important point is that projective noncommutative geometry does not require a topological space of points as a primitive object; the category carries the sheaf-theoretic information.

Proposition 2 (classical limit of the deformation model). Let $A_\theta = A_0[[\theta]]$ be a formal noncommutative coordinate algebra with specialization map $\pi: A_\theta \to A_0$ defined by $\pi(\sum_{n\geq 0} a_n\theta^n) = a_0$. Then π is a surjective C-algebra homomorphism from A_θ to A_0 when A_0 is given its ordinary product and A_θ is reduced modulo θ.

Proof. Surjectivity is immediate because any $a_0 \in A_0$ is the image of the constant series a_0. For multiplication, let $F = \sum f_n\theta^n$ and $G = \sum g_n\theta^n$. The zeroth-order term of $F *_\theta G$ is f_0g_0 because $*_\theta$ reduces to the ordinary product modulo θ. Hence $\pi(F *_\theta G) = f_0g_0 = \pi(F)\pi(G)$. The map is therefore a homomorphism after specialization modulo θ.

This result establishes the sense in which the classical Riemann surface is retained inside the deformation. The noncommutative object is not detached from the classical one; it is a thickened or deformed algebraic structure whose zeroth-order quotient is commutative. This is the algebraic foundation for comparing classical and noncommutative geometries within the same framework.

8. Module-Theoretic Interpretation

The passage from a commutative Riemann surface X to a noncommutative algebra Aθ changes the role of modules. In the commutative case, modules over coordinate rings can often be localized and interpreted as sheaves. When it is non commutative in the case the localization is more delicate and left right distinction is vital. A right Aθ-module M and a left Aθ-module N may behave differently because the equality am = ma has no meaning unless a compatible bimodule structure is imposed.

The category Mod(Aθ) of right modules over Aθ is the first replacement for a category of sheaves. When Aθ is graded, finitely generated graded modules category gr(Aθ) and the quotient qgr(Aθ) play the role of coherent sheaves. This framework is not merely academic. Recent research has shown that module categories retain useful geometric information such as regularity, duality, and categorical dimension in the context of quantum projective spaces, connected Z-algebras, and noncommutative hypersurfaces (Mori & Nyman, 2021; Mori & Ueyama, 2021; Mori & Ueyama, 2022).

Projective modules are particularly important. A finitely generated projective Aθ-module is a direct summand of finite free module. In commutative geometry, vector bundles correspond with finitely generated projective modules over suitable rings.

In noncommutative categories, they act as vector-bundle analogues. The usual pointwise fibres are replaced by idempotent and module-theoretic data. If e belongs to Mn(Aθ) and is idempotent, it follows that eAθn is a finitely generated projective right module. This is an algebraic substitute of a vector bundle on Xθ.

Bimodules communicate noncommutative Riemann surfaces. If Aθ and Bη are two noncommutative coordinate algebras, the Aθ-Bη-bimodule P can define a functor from Mod(Bη) to Mod(Aθ) by tensor product. When bipartite graphs satisfy finiteness and local generation conditions, this functor is an equivalence. So, bimodules are not auxiliary objects, they are morphisms in the categorical sense. The concept which is the essence of Morita equivalence, important for classification.

The viewpoint of the module-theoretic explains why noncommutative Riemann surface cannot be reduced to its underlying set of representations. Representations are useful, but the full module category has extension groups, torsion modules, projective modules, and derived information. For noncommutative geometries that resemble curves, coherent modules have a structure that is similar to that of classical curves. Burban and Drozd's (2022) treatment of noncommutative noetherian schemes and the case of reduced noncommutative curves makes it concrete.

9. Framework of Sheaf and Laws.

The sheaf-theoretic interpretation of noncommutative Riemann surfaces starts from the observation that sheaves already form a category in classical geometry. The structure of an object X in Space (X) may be understood using Sh (X), Coh (X), and QCoh (X).

Within the area of noncommutative geometry, one performs this symmetrically; rather than first constructing a point-set space and attaching sheaves, one construct a category that acts like a sheaf category and treats it as the geometry.

The category qgr(A) of a graded algebra A is obtained by eliminating all finite-dimensional torsion modules, while retaining projectives. This parallels the role of Proj in commutative algebraic geometry. The qgr(A) objects are not sheaves on a classical topological space (such as the spectrum of a ring), but they behave like coherent sheaves on a noncommutative projective scheme. In qgr(A), morphisms substitute sheaf morphisms, extensions replace cohomological relations, and the derived category of qgr(A) encodes deeper geometry. The importance of derived/categorical structures for recovering geometric information in noncommutative projective geometry is shown in the study of (Mizuno 2024a).

The category Cθ should be curve-like for a noncommutative Riemann surface Xθ. It should accept coherent objects, torsion-like objects, vector-bundle analogues and homological invariants, which distinguish dimension-1 from higher dimension geometry. If Aθ is a deformation of the commutative curve algebra, then specialization modulo θ should relate Cθ to Coh(X). The phrase noncommutative Riemann surface should not become too general: not every noncommutative algebra deserves a name; the category should remain curve-like in coherence.

The sheaf-theoretic framework solves the problem that noncommutative algebras do not have enough points. In commutative algebraic geometry, one associates to points prime ideals. However, prime ideal theory is often inadequate in the noncommutative setting. People usually consider using simple modules or point modules or fat point modules. Or, maybe, categorical points whose usefulness depends on the

situation. Research involving quantum projective planes and noncommutative conics reveals that it is possible to recover point-like and curve-like objects through the use of modules and projective categorical data (Hu et al. 2023; Itaba & Mori 2023).

Hence, the sheaf-theoretic replacement preserves geometry and is not a loss. There is a variation in the value where geometry encodes. Classical sheaves are associated with open sets, while associated with noncommutative sheaves are organized through modules, quotient categories, derived categories. The central message is that geometry is reconstituted through algebraic relations among objects, rather than by points.

10. Equivalence and classification of Morita.

The language of Morita equivalence provides a natural classification of noncommutative Riemann surfaces. An equivalent algebra is the one that has the same representation theory. In a geometric interpretation, this means that A and B define the same sheaf-like objects even though the algebras may not be isomorphic. This is more than a superficial analogy: in noncommutative geometry, categories of modules often carry the geometrical content which points and ordinary functions carry in the commutative case.

Proposition 3 (stabilisation matricielle). For any associative algebra A and any $n \geq 1$, we have that the matrix algebra $M_n(A)$ is Morita equivalent to A.

证。 Let $P = A^n$ be the column module, which we view as a left $M_n(A)$-module and a right A-module. Both the functor $\mathrm{Hom}_A(P, -)$ and the tensor functor $- \otimes_{M_n(A)} P$ provide inverse equivalences between suitable module categories. To say that P is a finitely generated projective generator is equivalent to saying that $\mathrm{End}_A(P) \cong M_n(A)$. Consequently, the module categories of $M_n(A)$ and A are equivalent.

This proposition shows why the isomorphism of algebras is too rigid. The algebras A and $M_n(A)$ may seem different but their module categories are equivalent. A geometric theory that classified noncommutative spaces solely by the isomorphism of algebras would incorrectly classify many matrix-stabilized forms of … as different geometries.

The Morita viewpoint gives a classification principle for noncommutative Riemann surfaces. Two deformation-model surfaces $X_\theta = (A_\theta, C_\theta, \pi_A)$ and $Y_\eta = (B_\eta, D_\eta, \pi_B)$ are algebraically equivalent if and only if C_θ and D_η are equivalent categories that preserve coherent objects, projective objects, and torsion subcategories, classical specialization, and further if two coherent sheaves are isomorphic if and only if their images are isomorphic under the functor of the respective categories. What this means in projective models

is that we can write abstractly qgr(Aθ)=qgr(Bη). In scheme-type models, it means equivalence between categories of quasi-coherent sheaves. The authors Burban and Drozd 2022 develop exactly this type of Morita-theoretic thinking for noncommutative noetherian schemes.

The Morita equivalence introduces the specification for the role of Z(A). The center is commutative and is often a shadow of the underlying classical space. Nonetheless, the center alone seldom dictates the noncommutative geometry. Two algebras may have related centres but different module categories or may be Morita equivalent in ways hidden from elementary ring invariants. The research conducted on quantum projective planes which are finite over their centers demonstrates the interaction of central finiteness and projective geometry (Itaba & Mori, 2023).

The classification of noncommutative Riemann surfaces should therefore be multi-tiered. At the first level is recorded the classical limit X. At the second tier, the deformation or graded coordinate algebra Aθ is being recorded. At the third level, we record the sheaf-like category Cθ. At level 4, taking Morita equivalences, one identifies derived equivalences. This stratification does not prefer the first alternative in that it does not take as a surface every noncommutative algebra, nor the second in that it does not define every point when there is no adequate point-set object.

11. Comparative Analysis

Table 1 compares the ordinary analytic, algebraic, and noncommutative descriptions relevant to the present paper. The comparison is conceptual, not empirical, and is intended to show how each classical component is replaced in the noncommutative model.

Table 1 Classical, Algebraic, and Noncommutative Models

Criterion	Classical Riemann surface	Algebraic curve	Noncommutative Riemann surface
Basic object	One-dimensional complex manifold	Smooth projective curve over C	Algebraic-categorical deformation object
Coordinate structure	Holomorphic and meromorphic functions	Commutative coordinate or homogeneous ring	Noncommutative algebra Aθ or graded algebra A
Multiplication	Pointwise and commutative	Commutative ring multiplication	Deformed product f ∗θ g, generally noncommutative

Sheaves	Sheaves over open sets	Coherent and quasi-coherent sheaves	Modules, qgr(A), QGr(A), or sheaf-like categories
Points	Analytic points	Prime ideals or closed points	Representations, point modules, simple modules, or categorical points
Morphisms	Holomorphic maps	Scheme morphisms and pullbacks	Functors, bimodules, equivalences, and categorical correspondences
Equivalence	Biholomorphism	Isomorphism of curves or equivalence of sheaf categories	Morita equivalence or derived/categorical equivalence
Classification emphasis	Genus, complex structure, moduli	Function field, divisors, sheaves, moduli	Deformation class, module category, center, Morita class
Main limitation	Requires point-set manifold	Requires commutative algebra	Requires categorical interpretation; point-set intuition is secondary

The table demonstrates that noncommutative Riemann surfaces cannot be obtained by merely changing a few local coordinates. The entire geometry language locus shifts. Noncommutative algebras take the place of coordinate rings; module categories take the place of sheaves; functors and bimodules take the place of morphisms; and classification is by Morita equivalence or categorical equivalence. This change matches the recent works on noncommutative schemes and noncommutative projective varieties (Burban & Drozd, 2022 Gammelgaard & Gyenge, 2024 Mizuno, 2024a).

The ineffectiveness of commutativity is also seen because no two sides are opposite sides of a coin. The Controlled Extension of Algebraic Curve Model is Noncommutative Model The legitimacy of the derived category, at least in relation to algebraic geometry, depends on retaining sufficient curve-like categorical structure to render sheaves, vector bundles, divisors and moduli spherical or manageable in transformed form.

12. Key Findings.

The first finding: Noncommutative Riemann surfaces should be defined by algebraic-categorical data and not just by pointwise noncommuting coordinates. The triple $(A\theta, C\theta, \pi)$ has the essential components, namely a noncommutative coordinate algebra, an associated sheaf-like category, and a classical specialization. This prevents any such ambiguity that treating a noncommutative algebra would allow for.

Further, the second finding is that deformation gives us a good measure of noncommutativity. The expression $f *\theta g = fg + \theta C1(f,g) + O(\theta^2)$ indicates that the first-order failure of commutativity is determined by the antisymmetric part of C1. Thus, the commutator $[f,g]\theta$ is the algebraic sign of the passage from a classical surface to a noncommutative one. Compact Riemann surfaces are complex one-dimensional, so a distinction must be made between holomorphic deformation and smooth or graded algebraic deformation.

A better way to reconstruct the sheaf-theoretic content of a noncommutative Riemann surface is through modules and quotients categories. This aligns with recent categorical approaches to quantum projective spaces and connected Z-algebras (Mori and Nyman 2021; Mori and Ueyama 2021). The present paper explores how the category qgr(A), when it exists, is the noncommutative analogue of coherent sheaves on a projective curve.

The fourth finding is that Morita equivalence is the correct first level classification principle. The condition of isomorphism of noncommutative algebras is too strict since particular matrix stabilizations and related algebraic transformations would preserve the module categories. Morita equivalence refers to the geometric similarity of sheaf-like categories. This was just recently reinforced by the Morita theory for Nonncommutative Noetherian Schemes (Burban & Drozd, 2022).

Noncommutative Riemann surfaces are linked to many mathematical areas including algebraic curves, deformation theory, noncommutative rings, category theory, projective geometry, and mathematical physics. Research on the noncommutative conics, quantum projective planes, noncommutative cubic surfaces, and noncommutative super surfaces demonstrates that noncommutative structures which are curve-like and surface-like are active objects in mathematics (Abdelgadir et al., 2024; Hu et al., 2023; Wang & Wu, 2022).

13. Discussion

This paper's algebraic analysis favors a middle-ground perspective of noncommutative Riemann surfaces. A noncommutative Riemann surface should not be thought of as an ordinary Riemann surface whose local coordinates do not commute. Such language may suggest this, but it does not suffice mathematically. The reliable definition which we shall use is categorical. This means that one starts with an algebraic or analytic structure that a Riemann surface supports. Remove or deform its coordinate algebra and study the resulting module category as carrier of geometry.

Due to the constraints imposed by dimension one, this view is especially important. In complex dimension one, triviality of holomorphic Poisson structures occurs in the ordinary sense of bivector. Thus, a naive

holomorphic star product cannot mimic the deformation quantization in higher dimensions. As a result, it motivates smooth symplectic deformation, noncommutative projective coordinate algebras, relevant orders over curves, and categorical structures. The noncommutative object is deemed valid not because it preserves every local analytic feature of a Riemann surface rather, it does so because it preserves a controlled algebraic and categorical relationship to the classical curve.

Scripts play a crucial role. In conventional geometry, it is possible to obtain significant information about a space from its sheaves. In the philosophy of noncommutative geometry, the category of modules is even more basic because the underlying points do not matter. Finitely generated projective modules behave (like) vector bundles; torsion modules behave (like) point-like or finite-support objects; while quotient categories (like qgr(A)) behave (like) coherent sheaves on projective schemes. Noncommutative Geometry – Professor S. K. S. S. Ramanujan 2 09-20-2023 18:27 2 65 The study of noncommutative conics and quantum projective planes shows that low dimensional noncommutative geometry can still retain “curve” properties, mostly when they are far away from commutative coordinate rings (Hu et al, 2023 and Itaba and Mori, 2023).

The meaning of classifications is changed by Morita equivalence. In commutative situations, one commonly expects that the isomorphism of the coordinate rings and isomorphism of spaces are closely related. In a noncommutative setting, this expectation needs to be relaxed. Matrix algebras, endomorphism algebras of progenerators and equivalent module categories represent the same noncommutative geometry. Classification by Morita equivalence therefore respects the categorical nature of the object better. Burban and Drozd (2022) demonstrate that this perspective leads to tangible conditions for noncommutative schemes and curves, not just philosophical considerations.

It further demonstrates why Riemann surfaces are noncommutative. It offers an algebraic language for the study of quantum surfaces, deformation phenomena, moduli of sheaves, and categorical forms of curve theory. While adding higher-order deformation terms or categorical replacements, they permit the classical curve to survive as a limit by preserving the structure of the singularities. In mathematical physics, such structures are relevant because observables are typically noncommutative; in algebraic geometry, they are relevant because they extend the geometric meaning of the rings, modules and derived categories.

At the same time, we need to be careful. A theory that is going to be published should not overclaim that one construction encapsulates all noncommutative Riemann surfaces. The expression can be understood better as a family of related algebraic models. The explicitness of the proposed model allows us to require a

coordinate algebra and a sheaf-like category with classical specialization. This imposes sufficient structure for comparative rigour without imposing a definition common to all that the existing literature can support.

14. Investigating Deficiencies and Future Possibilities

There are categories which are derived. The current paper primarily operates within the realm of module and quotient categories. Future studies should create derived categories of noncommutative Riemann surfaces and investigate the occasions when derived equivalence is stronger or more fitting than Morita equivalence. Mizuno, 2024a) Recent work in noncommutative projective geometry suggests that derived categories recover a lot of geometric data.

Another area of gap is concerned with noncommutative Teichmüller theory. The study of deformation of complex structure on Riemann surfaces is what classical Teichmüller theory is all about. A noncommutative version of something will have to identify deformation parameters not only for the complex structures but also for multiplication laws, module categories or equivalence classes of algebras. A clear distinction would need to be made between analytic deformation, formal algebraic deformation and categorical deformation.

There is a deficiency in spectral triples and metric information. The current paper is algebraic but noncommutative geometric also has analytical metric instruments. As part of future work, the authors say that it may be feasible to connect the model $(A\theta, C\theta, \pi)$ with spectral triples, Dirac operators, cyclic cohomology, and K-theory. Research in noncommutative super surfaces and noncommutative differential geometry shows that notions of metric and curvature can be constructed in noncommutative contexts that resemble surfaces.

The fourth gap pertains to moduli spaces. Recent studies on marked noncommutative cubic surfaces, noncommutative projective partial resolutions, and noncommutative difference-equation surfaces suggest that moduli of sheaves and moduli of noncommutative surfaces are becoming focused areas of study (Abdelgadir et al., 2024; Gammelgaard & Gyenge, 2024; Rains, 2025) . A theory of moduli for noncommutative Riemann surfaces would have to classify deformation algebras along with categorical equivalences and classical limits.

Fifth gap has to do with arithmetic and elliptic cases. Elliptic surface and noncommutative tori are promising links between noncommutative geometry and arithmetic geometry (Nikolaev, 2021). In the future research, it may be good idea to check if elliptic curves, quantum tori and projective curves that are noncommutative fit together in a single ‘algebraic’ theory of genus one noncommutative Riemann surfaces.

15. Conclusion; the end

We study noncommutative Riemann surfaces algebraically through deformation, sheaf-theoretic replacement, module categories, and Morita equivalence in this paper. The paper's main point is that we should consider a noncommutative Riemann surface rather as an algebraic-categorical object than as an ordinary point-set surface with noncommuting local coordinates. The structure $X\theta = (A\theta, C\theta, \pi)$ of the proposed model records a noncommutative coordinate algebra and an associated sheaf-like category, specializing, in the classical commutative surface.

The deforming product $*\theta$ will explain how the classical product is modified. While the first-order commutator will give the first obstruction to commutativity. The projective analogue of coherent sheaves is provided by qgr(A) and other quotient categories formed by sheaf categories. As similar categories of modules can define the same noncommuting Geometry although the algebras are not isomorphic, Morita equivalence provides the right classification principle.

A new type of Riemann surface, which has a non-commutative structure (that is, one that admits non-commutative coordinate rings) is introduced and analyzed. This is an extension of classical algebraic curves theory. The classical Riemann surface arises as a limit in this framework, while noncommutative algebra encodes new geometries. The theory that ensues serves as a solid basis for future research on noncommutative Teichmüller theory, derived categories, spectral triples, K-theory, and moduli spaces of noncommutative curves and surfaces.

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